Description of my Research

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0. Remarks for the non-expert

Almost all of my research has something to do with groups. I am particularly interested in Lie groups and their arithmetic subgroups.

(0.1) **Example.**

- We use $\mathrm{SL}(n,\mathbb{R})$ to denote the set of $n \times n$ real matrices of determinant one.
  1. This is a group under multiplication. (Inverses exist because the determinant is not allowed to be zero.)
  2. This is also a topological space (more precisely, a metric space) in a natural way. For example, we can define the distance between two matrices (a_{ij}) and (b_{ij}) to be $\max_{i,j} |a_{ij} - b_{i,j}|$. In fact, $\mathrm{SL}(n,\mathbb{R})$ is a *manifold* (of dimension $n^2 - 1$).
  3. The group operations of multiplication and taking inverses are continuous. (E.g., if matrix A_1 is close to A_2 and matrix B_1 is close to B_2, then the product A_1B_1 is close to A_2B_2.)

  These three observations, taken together, are precisely what it means to say that $\mathrm{SL}(n,\mathbb{R})$ is a *Lie group*.
- Roughly speaking, an *arithmetic group* is the integer points of a Lie group. (Technically, only certain Lie groups are allowed.) As a specific example, we use $\mathrm{SL}(n,\mathbb{Z})$ to denote the set of $n \times n$ integer matrices of determinant one. This is a subgroup of $\mathrm{SL}(n,\mathbb{R})$.

My main research interest is in the study of Lie groups, such as $\mathrm{SL}(n,\mathbb{R})$, and arithmetic groups, such as $\mathrm{SL}(n,\mathbb{Z})$.

I like the fact that the study of arithmetic groups combines algebra with other branches of mathematics, including topology, geometry, number theory, analysis, and dynamical systems. I am currently writing a book [M3] that I hope will make the fundamentals of the subject accessible to mathematicians in some of these other fields. Although I have a first draft of about 200 pages, I believe that I have only completed about 1/3 of the work on this project.

The vast majority of my research related to Lie groups can be put into one of four categories: (1) *Actions on the circle*, (2) *Superrigidity*, (3) *Tessellations of homogeneous spaces*, or (4) *Unipotent dynamics*. I have also written a number of papers on (5) *Hamiltonian cycles in Cayley graphs*, and I have a few (6) *Miscellaneous papers* that do not fit into the preceding categories. Later sections of this *Research Description* give a rather technical discussion of my main results in each of these six areas.

In these introductory remarks, I will provide a brief introduction to each of the six areas, and some indication of what I have done.

0.1. **Actions on the circle.** For any mathematical object X (whether it be a group, or a graph, or a Riemannian manifold, or anything else), an algebraist will want to understand the automorphism group of X. For example, if X is a topological space, then an algebraist will be interested in the algebraic structure of $\mathrm{Homeo}(X)$, the group of all homeomorphisms of X, that is, all the functions $f \colon X \to X$, such that

- f is both one-to-one and onto (so f has an inverse), and
- both f and f^{-1} are continuous.

In particular, for any group Γ, one may ask whether $\mathrm{Homeo}(X)$ contains Γ, or, more precisely, whether $\mathrm{Homeo}(X)$ contains a subgroup that is isomorphic to Γ. (In group-theoretic language, we are asking whether Γ has a faithful, continuous *action* on X.)

(0.2) **Assumption.** Let us consider only the case where X is a union of finitely many (disjoint) line segments. Even this easy-looking case turns out to be quite difficult.

(0.3) **Example.** If X is the union of n line segments, then it is easy to see that $\mathrm{Homeo}(X)$ contains the symmetric group S_n. Because any finite group is contained in some symmetric group, we conclude that if Γ is any finite group, then Γ is contained in $\mathrm{Homeo}(X)$, for some such X.

As a less obvious example, it turns out that if X is the union of 12 line segments, then $\mathrm{Homeo}(X)$ contains $\mathrm{SL}(2,\mathbb{Z})$. (This is a consequence of the fact that $\mathrm{SL}(2,\mathbb{Z})$ contains a free subgroup of index 12.) I proved that $\mathrm{SL}(3,\mathbb{Z})$ is quite different from $\mathrm{SL}(2,\mathbb{Z})$ in this regard:

(0.4) **Theorem** (cf. 1.1)**.** *If X is the union of any finite number of line segments, then* $\mathrm{Homeo}(X)$ *does* ***not*** *contain* $\mathrm{SL}(3,\mathbb{Z})$*.*

I proved this by using the following well-known algebraic characterization of the subgroups of $\mathrm{Homeo}(X)$:

(0.5) **Definition** [MR, Chap. VII]**.** A group Γ is *right orderable* if there exists a subset P of Γ, such that

(1) for all $g, h \in P$, we have $gh \in P$, and
(2) Γ is the **disjoint** union of P, P^{-1}, and $\{e\}$, where $P^{-1} = \{\, g^{-1} \mid g \in P \,\}$ and e is the identity element of Γ.

(0.6) **Example.** The additive group $(\mathbb{R}, +)$ is right orderable. (Let $P = \{x \mid x > 0\}$.)

In general, to say that Γ is right orderable means, for some subset P, that we can think of P as the set of "positive" elements of Γ and P^{-1} as the set of "negative" elements.

(0.7) **Proposition.** *If Γ is any countable group, then the following are equivalent:*

(1) *Γ is contained in* $\mathrm{Homeo}(X)$*, for some X that is the union of finitely many line segments.*
(2) *Some finite-index subgroup of Γ is right orderable.*

So I showed that no finite-index subgroup of $\mathrm{SL}(3,\mathbb{Z})$ is right orderable. The same is true for $\mathrm{SL}(n,\mathbb{Z})$, whenever $n \geq 3$, but, for many other arithmetic groups Γ, we still do not know whether Γ is right orderable or not. Several mathematicians, including me, are thinking about this problem.

Although the above discussion was written in terms of line segments, I also proved a version of my theorem (0.4) for the case where X is a circle (or a finite union of circles).

0.2. **Superrigidity.** My superrigidity theorems are group theoretic, but, as motivation, let us first discuss an analogous notion in combinatorial geometry.

(0.8) **Example.**

- Alice is given a collection of 3 sticks (line segments) S_1, S_2, S_3.
  - The two ends of S_1 are labelled A and B.
  - The two ends of S_2 are labelled B and C.
  - The two ends of S_3 are labelled A and C.
- She is told to construct a geometric object, by
  - gluing together all the points labelled A,
  - gluing together all the points labelled B, and

◦ gluing together all the points labelled C.

Of course, the result is a triangle.

It is important to note that if Alice takes the triangle apart, and reassembles it, then she is guaranteed to get back the same triangle (up to congruence), because of "side-side-side." This is what it means to say that the triangle is *superrigid* — any reassembled version of the triangle is congruent to the original one.

(0.9) **Example.** In contrast, suppose Alice is given the set of five sticks in Fig. 1, which can be assembled to form two triangles that are joined along an edge, as illustrated in either of the pictures in Fig. 2. Let us us call the resulting object a "hinge." The two hinges pictured in Fig. 2 are not congruent, so a hinge is not superrigid. Indeed, we can say that a hinge is not even *rigid*, because it admits a continuous family of deformations.

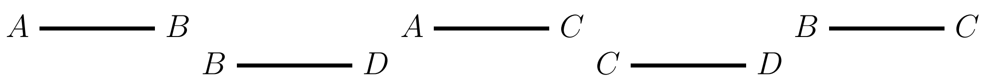

FIGURE 1. Five sticks that can be assembled into a hinge.

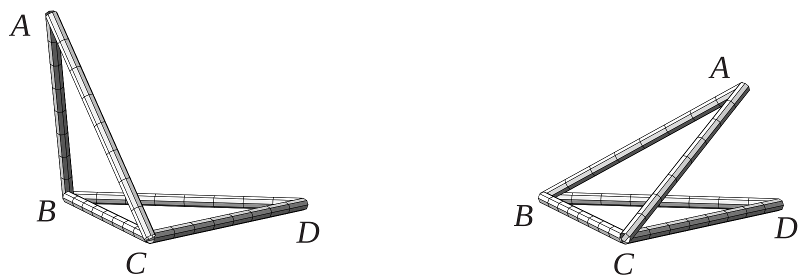

FIGURE 2. Two noncongruent objects that can be made from the same set of labelled sticks. Uncountably many different shapes can be made (corresponding to different angles between the two triangles), and we call each of them a "hinge."

(0.10) **Example.** Now suppose Alice is given the set of nine sticks (six long and three short) in Fig. 3. They can be assembled into either of two noncongruent objects, as illustrated in Fig. 4. Thus, these two objects are **not** superrigid. However, there is no continuous family of deformations, so they are rigid.

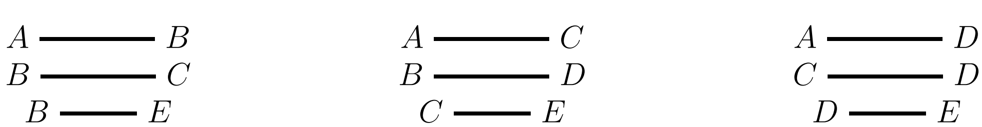

FIGURE 3. Nine sticks that make two tetrahedra joined on a face.

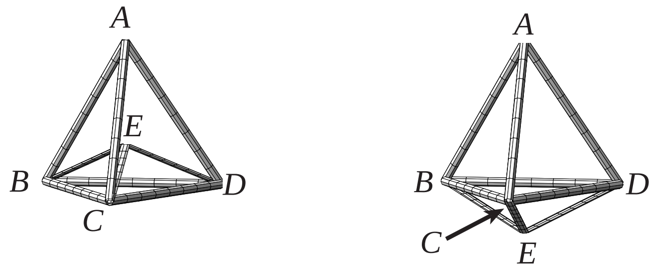

FIGURE 4. The small tetrahedron can be either inside or outside the large one.

(0.11) **Definition.** I will not give a formal definition here (cf. [GSS]), but the idea is that a stick figure is *superrigid* if its geometric structure (how it sits in 3-space, up to a rigid motion) is completely determined by the combinatorial data defined by its sticks. More precisely, if

- X is a superrigid stick figure in $\mathbb{R}^3$, and
- X' is any reassembled version of X, in any Euclidean space $\mathbb{R}^n$,

then there is an isometric embedding $f\colon \mathbb{R}^3 \hookrightarrow \mathbb{R}^n$, such that $f(X) = X'$.

The above is a geometer's version of superrigidity. My superrigidity theorems consider an analogous notion that is of interest to group theorists.

(0.12) **Definition.** Roughly speaking, if

- Γ is a *superrigid* subgroup of a Lie group G, and
- Γ' is any subgroup of any $\mathrm{SL}(n,\mathbb{R})$, such that Γ' is isomorphic to Γ,

then there is a homomorphism $f\colon G \to \mathrm{SL}(n,\mathbb{R})$, such that $f(\Gamma) = \Gamma'$.

Under the assumption that G is solvable (and connected), I gave simple conditions that determine whether any particular subgroup Γ is superrigid or not (see 2.2).

There is an obvious analogy between the above definitions (with the groups Γ, G, and $\mathrm{SL}(n,\mathbb{R})$ in the place of the geometric objects X, $\mathbb{R}^3$, and $\mathbb{R}^n$). Because the geometric version seems to be of intrinsic interest, this analogy should be seen as evidence that the algebraic version is also of interest. In fact, G. A. Margulis' deep theorem that lattices in most of the simple Lie groups are superrigid is of tremendous importance. (It implies, for example, the fundamental fact that, in most simple Lie groups, every lattice is an arithmetic subgroup.) My generalizations of this theorem are not nearly so important, but they do have applications. Also, the techniques I developed in proving the result can be used to attack other questions about the structure of solvable Lie groups.

0.3. **Tessellations of homogeneous spaces.** Tessellations (repeating patterns) in various geometric spaces are of tremendous importance, both in applications and in pure mathematics.

(0.13) **Example.**

- Tessellations of the Euclidean plane $\mathbb{R}^2$ are common designs on textiles, wallpaper and tiled floors (cf. Fig. 5).

- Some of M. C. Escher's artwork is based on tessellations of the hyperbolic plane $\mathcal{H}^2$ (cf. Fig. 5).
- The molecular structure of a crystalline substance (such as diamond) is a tessellation of the Euclidean 3-space $\mathbb{R}^3$.
- Arithmetic groups are the symmetry groups of tessellations of certain manifolds, called symmetric spaces.

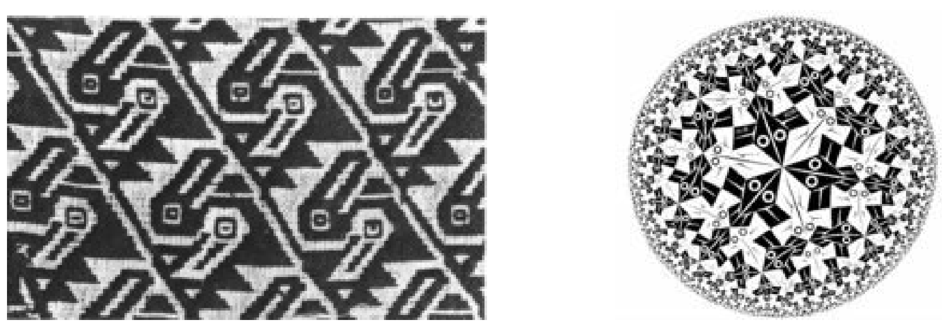

FIGURE 5. A fabric from Peru [GbS, p. 4] and Escher's "Circle Limit I" [Es, p. 22].

(0.14) **Definition** (Klein Erlanger Program)**.** A *geometric space* is a topological space X, together with a Lie group G, such that G acts transitively on X. That is, for all $x, y \in X$, there exists $g \in G$, such that $gx = y$.

We call G the *isometry group* of X.

(0.15) **Example.** The Euclidean space $\mathbb{R}^n$ is a geometric space, with G the group of all rigid motions of $\mathbb{R}^n$. Thus, the terminology "isometry group" agrees with the classical notion in this case.

Given any geometric space X, a geometer would like to know all of its tessellations. The first step is to decide whether there exist any tessellations at all. Hee Oh, Alessandra Iozzi, and I have written several papers on this existence question.

It has been known for at least twenty years that if a geometric space X has a tessellation, and its isometry group is $\mathrm{SL}(2, \mathbb{R})$, then X is rather trivial: either

- X is compact, or
- the stabilizer $\mathrm{Stab}_G(x)$ is compact, for every $x \in X$.

Oh, Iozzi and I proved the same result for $\mathrm{SL}(3, \mathbb{R})$ (see 3.4).

Furthermore, we found all the geometric spaces X, such that

- X has a tessellation, and
- the isometry group of X is either $\mathrm{SO}(2, 2n)$ or $\mathrm{SU}(2, 2n)$

(see 3.8 and 3.9). This includes infinite families of new examples of spaces that have tessellations.

0.4. **Unipotent dynamics.** The theory of dynamical systems studies the iterates $f^n = f \circ f \circ \cdots \circ f$ of a map f.

(0.16) **Example.**

- For any $\alpha \in \mathbb{R}$, we can define a *translation* $f_\alpha \colon \mathbb{R} \to \mathbb{R}$ by $f_\alpha(x) = x + \alpha$. The iterates of f_α are very easy to understand:

$$f_\alpha^n(x) = x + n\alpha,$$

so the points of $\mathbb{R}$ just move farther and farther to the right as n increases.
- To make this more interesting, consider f_α as a map on the quotient space $\mathbb{R}/\mathbb{Z}$ (a circle). If α is irrational, then it is well known that every orbit is dense; that is,

$$\{\, f_\alpha^n(x) \mid n \in \mathbb{Z} \,\} \text{ is dense in } \mathbb{R}/\mathbb{Z}, \text{ for every } x \in \mathbb{R}/\mathbb{Z}. \tag{0.17}$$

From (0.17), it is easy to see that translations are the only continuous maps that commute with f_α:

(0.18) *Remark.* Assume α is irrational. If $T \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is any continuous function, such that

$$T \circ f_\alpha = f_\alpha \circ T,$$

then T is a translation. That is, $T = f_\beta$, for some $\beta \in \mathbb{R}$.

This remark is an example of a "rigidity" result: it shows that the only answer to a certain question is the one that was obvious from the start. Namely, it is obvious that translations are examples of maps that commute with f_α, and the remark says that there are no others.

To obtain the conclusion that T is one of the obvious examples, we assumed that T is continuous. Because continuity is a topological notion, one says, for short, that f_α is *topologically rigid*. Although not so obvious, it has been known for more than fifty years that f_α is also *measurably rigid* – we need only assume that T is measurable, not continuous:

(0.19) **Proposition.** *Assume α is irrational. If $T \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is any Borel measurable function, such that*

$$T \circ f_\alpha = f_\alpha \circ T \text{ a.e.},$$

then T is a translation (a.e.).

Twenty years ago, M. Ratner proved a version of this theorem that:

- replaced the abelian group $\mathbb{R}$ with the much more interesting group $\mathrm{SL}(2,\mathbb{R})$,
- replaced the subgroup $\mathbb{Z}$ of $\mathbb{R}$ with any discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, such that $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is compact,
- replaced the real number α with the "unipotent" matrix $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and
- replaced the translation f_α with the translation $f_u \colon \mathrm{SL}(2,\mathbb{R})/\Gamma \to \mathrm{SL}(2,\mathbb{R})/\Gamma$, defined by $f_u(x\Gamma) = ux\Gamma$.

(Note that 1 is the only eigenvalue of the matrix u; that is the definition of *unipotent*.)

(0.20) **Theorem** (Ratner Rigidity Theorem)**.** *If $T \colon \mathrm{SL}(2,\mathbb{R})/\Gamma \to \mathrm{SL}(2,\mathbb{R})/\Gamma$ is any Borel measurable function, such that*

$$T \circ f_u = f_u \circ T \text{ a.e.},$$

then T is a translation (a.e.).

A few years later, in my Ph.D. thesis, I proved the appropriate generalization of this theorem to the situation where $\mathrm{SL}(2,\mathbb{R})$ is replaced with any (connected) Lie group G, and u is a unipotent element of G.

Later, in 1988, M. Ratner proved a tremendous generalization that subsumed my theorem and much other work on the subject. The general theorem is a remarkable result, with important applications in number theory and dynamics. Many people have found Ratner's theorem to be useful in their work, but it seems that few understand the proof. I am currently writing an expository paper [M2] that I hope will make the proof accessible to a wider audience of number theorists, dynamicists, Lie theorists , and geometers.

0.5. **Hamiltonian cycles in Cayley graphs.** It is not difficult to see that if two graphs X and Y each have a hamiltonian cycle, then their cartesian product $X \times Y$ also has a hamiltonian cycle.

(0.21) **Definition.** The *cartesian product* of X and Y is defined as follows:

- The vertex set $V(X \times Y)$ is $V(X) \times V(Y)$; and
- there is an edge joining (x, y) to (x', y') if either
  - $x = x'$ and there is an edge (in Y) joining y to y', or
  - $y = y'$ and there is an edge (in X) joining x to x'.

This is not true for digraphs. (For example, suppose X and Y are directed cycles. If the length of X is relatively prime to the length of Y, then $X \times Y$ does not have a hamiltonian cycle.) On the other hand, S. Curran and I [CW] showed that it **is** true for products of three (or more) digraphs.

(0.22) **Theorem** (Curran and Witte)**.** *If X, Y, and Z are digraphs, and each of the three has a hamiltonian cycle, then the cartesian product $X \times Y \times Z$ also has a hamiltonian cycle.*

The proof easily reduces to the case where X, Y, and Z are directed cycles. Then the vertex set of $X \times Y \times Z$ can be identified with the abelian group $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$, where a, b, and c are the lengths of X, Y, and Z. This allows one to apply group-theoretic techniques, such as modding out a subgroup to form a quotient group. It also brings the problem into the realm of so-called "Cayley digraphs."

Circulants are the most basic examples of Cayley digraphs.

(0.23) **Definition.** For any subset S of the cyclic group $\mathbb{Z}_n$, we define $\operatorname{Circ}(n; S)$, the *circulant digraph* of order n with generating set S, as follows:

- the vertices of $\operatorname{Circ}(n; S)$ are the elements of the cyclic group $\mathbb{Z}_n$, and
- for $v \in \mathbb{Z}_n$ and $s \in S$, there is a directed edge from v to $v + s$.

Loops are of no interest to us, so I will always assume $0 \notin S$. See Fig. 6 for two examples.

If $\#S = 1$ and $\operatorname{Circ}(n; S)$ is connected, then it is obvious that $\operatorname{Circ}(n; S)$ has a hamiltonian cycle. More than 50 years ago, R. A. Rankin realized that there are (connected) circulants with $\#S = 2$ that do not have a hamiltonian cycle, but I was apparently the first to discover such an example with $\#S = 3$ (namely, $\operatorname{Circ}(12; 3, 4, 6)$, pictured in Fig. 6). Recently, S. Locke and I showed that there are infinitely many of them [LW2].

In general, a *Cayley digraph* $\operatorname{Cay}(G; S)$ is defined as in Defn. 0.23, but with any (finite) group G in the place of $\mathbb{Z}_n$. One of my results [W2] shows that if $\#G$ is any p-group (that is, if $\#G$ is a power of a prime number), then $\operatorname{Cay}(G; S)$ has a hamiltonian cycle. (This theorem does *not* assume that G is abelian.) Unfortunately, however, we are still nowhere close to a complete answer to the question of Cayley digraphs have hamiltonian cycles, and which do not.

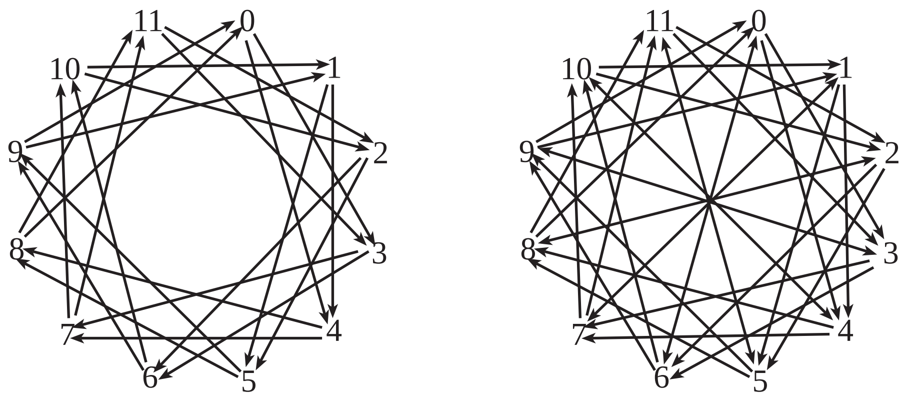

FIGURE 6. The circulant digraphs Circ(12; 3, 4) and Circ(12; 3, 4, 6). Neither of these has a hamiltonian cycle.

0.6. **Miscellaneous work.** I enjoy talking to people about their research, and I am especially pleased when I can provide helpful comments. Two of my recent papers [W13, M1] came out of questions from Scot Adams, and do not fit into any of the categories described above.

1. Actions on the circle

It is well known that the arithmetic group $\mathrm{PSL}(2,\mathbb{Z})$ acts faithfully on the unit circle (by linear-fractional transformations). Furthermore, some finite-index subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ are free, which means that they are able to act on every space. In contrast, I [W6] proved that finite-index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ do not behave at all like this.

(1.1) **Theorem** (Witte). *If Γ is a finite-index subgroup of $\mathrm{SL}(3,\mathbb{Z})$, then Γ does not have a faithful, continuous action on any connected 1-manifold (that is, on the circle S^1 or the real line $\mathbb{R}$).*

In fact, any continuous action of Γ on S^1 or on $\mathbb{R}$ must factor through a finite quotient of Γ. More generally, the same is true if Γ is any arithmetic subgroup of any simple algebraic $\mathbb{Q}$-group G with $\mathbb{Q}\text{-rank}(G) \geq 2$.

My theorem led to the following conjecture.

(1.2) **Conjecture.** The conclusion of Thm. 1.1 holds if Γ is any lattice in any connected, simple, linear Lie group of real rank at least two.

This conjecture is still open, but, for actions on the circle, É. Ghys [Gh] made important progress, including a complete solution of the differentiable case.

(1.3) **Theorem** (Ghys). *If Γ is a lattice in a connected, simple Lie group G, such that $\mathbb{R}\text{-rank}\, G \geq 2$, then*

(1) *every continuous action of Γ on the circle has a finite orbit, and*
(2) *Γ does not have a faithful, C^1 action on the circle.*

Under the additional assumption that $H^2(\Gamma;\mathbb{R}) = 0$ (and in many other cases), the conclusion of Ghys' theorem was also proved by M. Burger and N. Monod [BM1, BM2], as a consequence of vanishing theorems for bounded cohomology. The results of Burger and Monod also apply to the setting where $\mathbb{R}$ is replaced by other local fields; for example, Γ could be an S-arithmetic group.

1.1. **Algebraic formulation.** Although the statement of Thm. 1.1 refers to continuity and manifolds, which makes it seem to be a topological result, it can be restated in a purely algebraic form, which is how I proved it.

(1.4) **Definition** [MR, Chap. VII]**.** Let Γ be a group that is equipped with a total order $<$. We say Γ is *right ordered* if $a < b \Rightarrow ac < bc$ for all $a, b, c \in \Gamma$.

A group is *right orderable* if there exists an order relation under which the group is right ordered.

The following two simple lemmas establish the connection between right orderability and actions on $\mathbb{R}$ or S^1.

(1.5) **Lemma.** *If a group Γ acts faithfully on $\mathbb{R}$, by orientation-preserving homeomorphisms, then it is right orderable. Conversely, if a countable group is right orderable, then it has faithful actions on $\mathbb{R}$ and S^1 by orientation-preserving homeomorphisms.*

(1.6) **Lemma.** *Any faithful, orientation-preserving action of a group Γ on S^1 lifts to a faithful, orientation-preserving action of some central extension of Γ on $\mathbb{R}$.*

To establish Thm. 1.1, I proved (by purely algebraic methods) that Γ is not right orderable (and that no central extension of Γ is right orderable, either).

I am interested in pursuing other algebraic conditions that imply the conjecture. For example, Conj. 1.2 is closely related to a conjecture in the mainstream of current research on arithmetic groups. Namely, consider the case where Γ is an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group G with $\mathbb{Q}\text{-rank}(G) = 1$ and $\mathbb{R}\text{-rank}(G) \geq 2$. Platonov has conjectured that Γ has bounded generation (see [PR, p. 578]). In the case under consideration, it seems to be accepted that this bounded generation should be by unipotent subgroups. In most cases, it is not difficult to use this bounded generation (by unipotents) to show that Γ is not right orderable. I am working with L. Lifschitz to to settle the remaining cases. We have not yet succeeded in general (the $\mathbb{Q}$-rank-one lattices in $\mathrm{SL}(3,\mathbb{R})$ are the main stumbling block), and this will not be a major result, but it seems to be a worthwhile project.

I would also like to find a purely algebraic proof of Ghys's Theorem (1.3), or, at least, of the cases obtained by Burger and Monod. "Products of similar matrices," which I now describe, provide one possible approach.

If K is an infinite field, then it is well known that $\mathrm{SL}(n,K)$ is simple, modulo the scalar matrices. More precisely, if $A \in \mathrm{SL}(n,K)$, and A is not a scalar matrix, then every element of $\mathrm{SL}(n,K)$ can be written as a product of matrices that are conjugate to either A or A^{-1}. I [W11] showed that the inverse is superfluous:

(1.7) **Theorem.** *Let $A \in \mathrm{SL}(n,K)$, where K is a field, and assume that either $n > 2$ or $\#K > 3$. If A is not a scalar matrix, then every element of $\mathrm{SL}(n,K)$ is a product of matrices conjugate to A.*

In the same paper, I proved an analogous result for symplectic groups and special orthogonal groups, and for all semisimple algebraic groups over fields of characteristic 0 that are either local or algebraically closed. This leads to the following conjecture.

(1.8) **Conjecture** [W11]**.** Let G be a semisimple algebraic group over a field K of characteristic zero. Then, for every $x, y \in G_K$, either y is a product of conjugates of x, or there is a normal subgroup of G_K that contains x but does not contain y.

(1.9) *Remark.* The assertion of the conjecture is equivalent to the assertion that G_K is not partially orderable.

The following proposition establishes the connection with Ghys's Theorem. Hypothesis 1 of the proposition is automatically satisfied under the conditions of Conj. 1.2. Hypothesis 2 is the assumption of Burger and Monod. Hypothesis 3 is precisely the condition considered in the study of products of similar matrices.

(1.10) **Proposition.** *Let Γ be a lattice in a connected, semisimple Lie group G with finite center. If*

(1) $\Gamma'/[\Gamma',\Gamma']$ *is finite, for every finite-index subgroup Γ' of Γ;*
(2) $H^2(\Gamma;\mathbb{R}) = 0$*; and*
(3) *for every $x, y \in \Gamma$, either y is a product of conjugates of x, or there is a normal subgroup of Γ that contains x but does not contain y,*

then Γ does not have a faithful, C^1 action on the circle.

Unfortunately, my results on products of similar matrices are not complete even for groups of K-points, and the proposition requires an understanding of the $\mathbb{Z}$-points, which will, of course, be much more difficult to achieve.

1.2. S**-arithmetic groups.** R. J. Zimmer and I [WZ1] simplified one part of Ghys's proof. This allowed us to generalize his theorem to the case where Γ is an S-arithmetic group, without the assumption needed by Burger and Monod.

(1.11) **Theorem** (Witte and Zimmer)**.** *Let*

- *E be a global field;*
- *S be a nonempty, finite set of places of E, including all of the infinite places;*
- *$\mathbf{G}$ be a connected, almost simple algebraic group over E;*
- *$\mathcal{O}(S)$ be the ring of S-integers in E; and*
- *Γ be a finite-index subgroup of $\mathbf{G}\big(\mathcal{O}(S)\big)$.*

Assume

(a) $\sum_{s\in S} E_s\text{-rank}(\mathbf{G}) \geq 2$*; and*
(b) *for each archimedean $s \in S$, there is no continuous homomorphism from $\mathbf{G}(E_s)^\circ$ onto* $\mathrm{PSL}(2,\mathbb{R})$.

Then the conclusions of Ghys's Theorem hold.

1.3. **Actions on circle bundles.** R. J. Zimmer and I generalized Theorem 1.3 to the context of semisimple Lie group actions on circle bundles, or, more generally, $\mathrm{Diff}^1(\mathbb{T})$-valued Borel cocycles for ergodic actions of G.

(1.12) **Example.** Let

- H be a connected, semisimple Lie group;
- Γ be a torsion-free, cocompact lattice in H;
- T be a subgroup of H that is isomorphic to the circle group $\mathbb{T}$;
- G be a closed subgroup of H that centralizes T and acts ergodically on H/Γ; and
- $M = T\backslash H/\Gamma$.

Because Γ is torsion free and cocompact, we know that M is a compact manifold. Because G centralizes T, the action of G by translation on H/Γ factors through to an action on M; we see that H/Γ is a principal $\mathbb{T}$-bundle over M, and G acts on H/Γ by bundle automorphisms.

Thus, there is a Borel cocycle $\alpha\colon G\times M \to \mathbb{T}$, such that the action of G on H/Γ is isomorphic to the skew product $M \times_\alpha \mathbb{T}$. By assumption, the action of G on H/Γ is ergodic, so, if β is any cocycle cohomologous to α, then $M \times_\beta \mathbb{T}$ must be ergodic. Therefore, the image of β cannot be contained in any finite group of transformations of $\mathbb{T}$.

These examples show that there can be nontrivial cocycles into $\operatorname{Isom}(\mathbb{T})$, the isometry group of $\mathbb{T}$. If G has Kazhdan's property (T), then R. J. Zimmer and I [WZ1] showed, for a quite general class of G-actions, that every cocycle into $\operatorname{Diff}^1(\mathbb{T})$ is cohomologous to a cocycle into $\operatorname{Isom}(\mathbb{T})$. In more geometric terms, this states that if G acts on a circle bundle, preserving a probability measure on the base, then there is a G-invariant probability measure on the bundle.

1.4. **Actions on codimension-one foliations.** Theorem 1.3 provides examples of finitely generated groups Γ, such that every continuous action of Γ on the circle has a finite orbit. R. Feres and I [FW] showed, in many situations, that any foliation-preserving action of such a group on a codimension-one foliation must have a finite-index subgroup that fixes a leaf.

(1.13) **Assumption.** Let

- M be a closed, connected, smooth manifold;
- $\mathcal{F}$ be a transversely oriented, codimension-one, C^2 foliation of M; and
- Γ be a finitely generated group, such that every homomorphism from a finite-index subgroup of Γ into the group of homeomorphisms of the circle has a finite orbit on the circle.

(1.14) **Definition.** An action of Γ on M is *foliation-preserving* if $\gamma(L)$ is a leaf of $\mathcal{F}$, for every $\gamma \in \Gamma$ and every leaf L of $\mathcal{F}$.

Fix a foliation-preserving action of Γ on M.

(1.15) **Theorem** (Feres and Witte)**.** *If $\mathcal{F}$ has a closed leaf, then some closed leaf of $\mathcal{F}$ is fixed by a finite-index subgroup of Γ.*

(1.16) **Corollary** (Feres and Witte)**.** *If $\mathcal{F}$ admits a bounded transverse invariant measure, then it also admits a transverse invariant measure μ that is invariant under Γ, and each leaf in the support of μ is fixed by some finite-index subgroup of Γ.*

The next theorem provides a class of foliations on which some finite-index subgroup of Γ must fix all of the leaves.

(1.17) **Theorem** (Feres and Witte)**.** *If the germinal holonomy groups of all the non-compact leaves of $\mathcal{F}$ are trivial, then some finite-index subgroup Γ' of Γ fixes every leaf of $\mathcal{F}$.*

In particular, the conclusion holds if the non-compact leaves of $\mathcal{F}$ are simply connected.

2. Superrigidity

The Mostow Rigidity Theorem [Ms], Margulis Superrigidity Theorem [Mr2], and Zimmer Cocycle Superrigidity Theorem [Zi1] concern semisimple Lie groups. I [W7], [W12] proved analogues of these results for Lie groups that are not semisimple.

2.1. **Superrigid subgroups of solvable Lie groups.**

(2.1) **Definition.** A closed subgroup Γ of a **solvable** Lie group G is *superrigid* if, for every representation $\pi\colon \Gamma \to \mathrm{GL}_n(\mathbb{R})$,

- the restriction of π to some finite-index subgroup of Γ extends to a continuous representation $\hat\pi$ of G,
- such that $\hat\pi(G)$ is contained in the Zariski closure of $\pi(\Gamma)$.

I [W7, W9] obtained a complete characterization of the superrigid subgroups of solvable groups:

(2.2) **Theorem** (Witte)**.** *A closed subgroup Γ of a simply connected, solvable Lie group G is superrigid iff there exist*

- *a finite-index subgroup Γ' of Γ, and*
- *a semidirect-product decomposition $G = A \rtimes B$,*

such that

(1) $\Gamma' \subset B$,
(2) $\mathrm{Ad}_B \Gamma'$ *has the same Zariski closure as* $\mathrm{Ad}_B B$*, and*
(3) *the closure of* $[\Gamma', \Gamma']$ *is a finite-index subgroup of* $[B, B] \cap \Gamma'$.

For the case where Γ is a lattice, an elementary proof appears in [W14]. In this case, I [W7] also proved a somewhat more complicated version of the theorem that provides a quite detailed explanation of the obstructions that may force us to pass to a finite-index subgroup of Γ before obtaining an extension of π. (There are only three such obstructions.)

2.2. **Superrigidity of lattices in general Lie groups.** Let Γ be a lattice in a connected Lie group G. I [W7] showed that the question of whether Γ is superrigid in G reduces to the same question about a lattice in the maximal semisimple quotient of G. (When G is not solvable, Defn. 2.1 should be altered to require that the Zariski closure of $\pi(\Gamma)$ has no nontrivial, connected, compact, semisimple, normal subgroups.) The idea of the proof is that lattices in solvable groups are superrigid (Thm. 2.2), which means that the radical of G is under control, so all that remains is the semisimple part of G.

(2.3) **Theorem** (Witte)**.** *Let Γ be a lattice in an algebraically simply-connected Lie group G, and let C be the unique maximal connected, compact, semisimple, normal subgroup of G. Then Γ is superrigid iff*

- *the Zariski closure of* $\mathrm{Ad}_G(C\Gamma)$ *is equal to the Zariski closure of* $\mathrm{Ad}\, G$*, and*
- *the image of Γ in $G/(C \operatorname{Rad} G)$ is a superrigid lattice.*

The Margulis Superrigidity Theorem [Mr2, Thm. IX.5.12(ii), p. 327] asserts that irreducible lattices in semisimple groups of higher $\mathbb{R}$-rank are superrigid, and K. Corlette [Co] extended Margulis' result to some semisimple groups of real rank 1. (It is known that some of the lattices in other groups of real rank 1 are not superrigid.) If the experts can finish the study of semisimple groups of real rank 1, then we will have an essentially complete understanding of (archimedean) superrigidity for all Lie groups.

2.3. **Mostow Rigidity for solvable Lie groups.** The following classical theorem of Malcev [Mcv, Thm. 5] is a Mostow Rigidity Theorem for lattices in nilpotent Lie groups.

(2.4) **Theorem** (Malcev)**.** *Suppose Γ_1 and Γ_2 are lattices in simply connected nilpotent Lie groups G_1 and G_2. Then any isomorphism from Γ_1 to Γ_2 extends to an isomorphism from G_1 to G_2.*

Unfortunately, Malcev's theorem does not generalize in the obvious way to the class of all solvable groups. (Starkov [St] has made a detailed study of the case where $G_1 = G_2$.) However, I [W7] showed that Malcev's theorem does generalize if we allow "crossed isomorphisms" instead of only isomorphisms.

(2.5) **Definition** (cf. [CE, p. 168])**.** Let G_1 and G_2 be groups. A bijection $\phi\colon G_1 \to G_2$ is a *crossed isomorphism* if there is some homomorphism $\sigma\colon G_1 \to \operatorname{Aut} G_2$, such that the function

$$\sigma \times \phi\colon G_1 \to \operatorname{Aut} G_2 \ltimes G_2 \colon g \mapsto \bigl(g^\sigma, g^\phi\bigr)$$

is a homomorphism.

For a subgroup Γ_1 of G_1, we say that a crossed isomorphism ϕ is Γ_1*-equivariant* if $(g\gamma)^\phi = g^\phi \gamma^\phi$, for every $g \in G_1$ and $\gamma \in \Gamma_1$.

(2.6) **Theorem** (Witte)**.** *Let Γ_1 and Γ_2 be lattices in simply connected, solvable Lie groups G_1 and G_2. Assume $\overline{\operatorname{Ad}_{G_1} \Gamma_1} = \overline{\operatorname{Ad} G_1}$. Then any isomorphism from Γ_1 to Γ_2 extends to a Γ_1-equivariant crossed isomorphism ψ from G_1 to G_2.*

I also proved a rigidity theorem that does not require any assumption on the Zariski closures, but it is necessary to allow twisting in *both G_1 and G_2*. Thus, in general, ψ is a *doubly-crossed* isomorphism, instead of a *crossed* isomorphism. In spirit, this should be a corollary of my superrigidity theorem (2.2), but there are serious technical problems, because the conclusion of the rigidity theorem does not allow us to pass to finite-index subgroups of the lattices.

2.4. **Application to foliations of solvmanifolds.** Let Γ_1 be a lattice in a simply connected, solvable Lie group G_1. Any connected Lie subgroup X_1 of G_1 acts by translations on the homogeneous space $\Gamma_1\backslash G_1$; the orbits of this action are the leaves of a foliation $\mathcal{F}_1$ of $\Gamma_1\backslash G_1$. We call this *the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1*. Now suppose Γ_2 is a lattice in some other simply connected, solvable Lie group G_2, and that X_2 is a connected Lie subgroup of G_2, with corresponding foliation $\mathcal{F}_2$ of $\Gamma_2\backslash G_2$.

- It is natural to ask whether the foliation of $\Gamma_1\backslash G_1$ by cosets of X_1 is topologically equivalent to the foliation of $\Gamma_2\backslash G_2$ by cosets of X_2,
- or, more generally, whether there is a continuous map f from $\Gamma_1\backslash G_1$ to $\Gamma_2\backslash G_2$ whose restriction to each leaf of $\mathcal{F}_1$ is a covering map onto a leaf of $\mathcal{F}_2$. If so, it is of interest to know all the possible maps f.

If we assume that $\mathcal{F}_1$ has a dense leaf, and make certain minor technical technical assumptions on the lattices Γ_1 and Γ_2, then H. Bernstein and I [BW] showed that f must be a composition of maps of two basic types: a homeomorphism of $\Gamma_1\backslash G_1$ that takes each leaf of $\mathcal{F}_1$ to itself, and a doubly-crossed affine map.

This result had been proved in some special cases by D. Benardete [Bdt] and me [W4, Thm. 5.1]. Bernstein and I used the same basic methods as in [Bdt] and [W4], but my Mostow Rigidity Theorem for solvable Lie groups (see §2.3) allowed us to apply Benardete's method to a general solvable Lie group (with some additional technical complications).

The previous work of D. Benardete [Bdt] and D. Witte [W4] in in this area required G_1 and G_2 to be either solvable (as is assumed above) or semisimple. This is because Benardete's method (the basis of all of this work) relies on the Mostow Rigidity Theorem, which was only known in these cases. By applying my general superrigidity theorem (see Thm. 2.3), Bernstein and I applied Benardete's method to many groups that are neither solvable nor semisimple. However, unlike our work in the solvable case, our results in this general setting are not at all definitive, because we imposed severe restrictions on the subgroups X_1 and X_2. (However, the restrictions are always satisfied if X_1 and X_2 are one-dimensional.) We did not attempt to push these methods to their limit, because it seems clear that new ideas will be needed to settle the general case.

2.5. **Superrigidity of S-arithmetic groups.** Let $\mathbb{G}$ be a solvable linear algebraic group defined over $\mathbb{Q}$. My superrigidity theorem (2.2) implies that if the arithmetic subgroup $\mathbb{G}_{\mathbb{Z}}$ is Zariski dense, then it is a superrigid lattice in $\mathbb{G}_{\mathbb{R}}$. I [W8] proved an appropriate generalization of this statement for S-arithmetic subgroups, in place of arithmetic subgroups. Furthermore, as in (2.3), I obtained a superrigidity theorem for many non-solvable S-arithmetic groups by combining this result with the Margulis Superrigidity Theorem.

2.6. **Cocycle superrigidity for non-semisimple groups.** Suppose L is a semisimple Levi subgroup of a connected Lie group G, X is a Borel G-space with finite invariant measure, and $\alpha\colon X \times G \to \mathrm{GL}(n,\mathbb{R})$ is a Borel cocycle. Assume L has finite center, and that the real rank of every simple factor of L is at least two.

If L is ergodic on X, the Zimmer Cocycle Superrigidity Theorem [Zi1] often shows that (modulo a coboundary, a compact group and sets of measure zero), the restriction of α to L must be a homomorphism. That is, we may assume there is a continuous group homomorphism $\sigma\colon L \to \mathrm{GL}(n,\mathbb{R})$, such that $\alpha(\cdot, l) = \sigma(l)$, for all $l \in L$.

In this case, I [W12] proved that, after passing to a finite cover of X, the cocycle α is cohomologous to a homomorphism (modulo a compact group) on all of G, not just on L. That is, σ extends to a homomorphism $\tau\colon G \to \mathrm{GL}(n,\mathbb{R})$, such that, after replacing α by a cohomologous cycle, we have $\alpha(\cdot, g) = \tau(g)$, for all $g \in G$.

2.7. **Actions on compact principal bundles.** Let $G = \mathrm{SL}(n,\mathbb{R})$ (or, more generally, let G be a connected, noncompact, simple Lie group). For any compact Lie group K, it is easy to find a compact manifold M, such that there is a volume-preserving, connection-preserving, ergodic action of G on some smooth, principal K-bundle P over M.

(2.7) **Question.** Can M can be chosen independent of K?

R. J. Zimmer and I [WZ2] showed that if $M = H/\Lambda$ is a homogeneous space, and the action of G on M is by translations, then P must also be a homogeneous space H'/Λ'. Consequently, there is a strong restriction on the groups K that can arise over this particular M.

It would be very interesting to remove the assumption that the action preserves a connection. This might be a first step toward a version of the Zimmer Cocycle Superrigidity Theorem that would apply to cocycles into compact groups.

2.8. **Rigidity of some characteristic-p nillattices.** L. Lifschitz and I [LiW] looked for a version of the Mostow Rigidity Theorem for unipotent groups over nonarchimedean local fields, instead of $\mathbb{R}$. It is well known that if $\mathbb{G}$ is a unipotent algebraic group over a nonarchimedean local field L of characteristic zero, then the group $\mathbb{G}(L)$ of L-points of $\mathbb{G}$ has no nontrivial discrete subgroups. (For example, $\mathbb{Z}$ is not discrete in the p-adic field $\mathbb{Q}_p$.) Thus the case of characteristic zero is not of interest in this setting; we considered only local fields of positive characteristic.

(2.8) **Definition.** A discrete subgroup Γ of a topological group G is *automorphism rigid* in G if every automorphism of Γ virtually extends to a virtual automorphism of G.

Although it is easy to prove automorphism rigidity for abelian groups, it seems to be surprisingly difficult to prove for nonabelian groups. We do not have a general theory, and we do not have enough evidence to support a specific conjecture, but we obtained examples that suggest there may be mild conditions that imply arithmetic lattices are automorphism rigid.

(2.9) **Definition** cf. [BS, Ex. 9.2]**.** Let

$$G_2 = \left\{ \begin{pmatrix} 1 & y^p & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; y, z \in \mathbb{F}_p((t)) \right\}.$$

So G_2 is a two-dimensional, nonabelian, unipotent group over the local field $\mathbb{F}_p((t))$. Any subgroup of G_2 commensurable with $G_2 \cap \mathrm{GL}_3\big(\mathbb{F}_p[t^{-1}]\big)$ is an *arithmetic lattice.*

(2.10) **Theorem** (Lifschitz and Witte)**.** *If $p > 2$, then any arithmetic lattice Γ in G_2 is automorphism rigid.*

We do not know whether Theorem 2.10 remains true in the exceptional case $p = 2$.

(2.11) *Remark.* We also proved a similar result for arithmetic lattices in Heisenberg groups, but that was much easier.

We obtained Theorem 2.10 as an easy consequence of a result that explicitly describes every automorphism of Γ. An understanding of automorphism rigidity for general unipotent groups will require a method to prove Theorem 2.10 directly, not as a corollary of a classification of all automorphisms. This was the approach used in my work on superrigidity of lattices in solvable Lie groups (2.2), but the methods used there seem to be worthless in characteristic p. There are two key problems: first, Γ is not finitely generated in characteristic p; second, the extension to an automorphism of G is not unique—this makes it very difficult to argue by induction on the nilpotence class of G.

3. Tessellations of homogeneous spaces

(3.1) **Definition.** Let

- G be a Zariski-connected, almost simple, linear Lie group and
- H be a closed, connected, subgroup of G.

We say that the homogeneous space G/H has a *tessellation* if there is a discrete subgroup Γ of G, such that

$$\Gamma \backslash G/H \text{ is compact} \qquad \text{and} \qquad \Gamma \text{ acts properly on } G/H.$$

(Alternatively, one could say that G/H has a *compact Clifford-Klein form*, or simply that G/H has a *compact quotient.*)

Various homogeneous spaces of various simple groups have been considered by several authors, including Benoist [Bst], Benoist-Labourie [BL], Margulis [Mr3] and Zimmer [Zi2], but the situation in general is far from understood. (See the surveys [Kb2, La, Kb3].)

(3.2) **Example.** There are two classical cases in which G/H is well known to have a tessellation.

- If G/H is compact, then we may let $\Gamma = e$.
- If H is compact, then we may let Γ be any cocompact lattice in G. (A. Borel [Bo] proved that every almost connected, simple Lie group has a cocompact lattice.)

Thus, the existence of a tessellation is an interesting question only when neither H nor G/H is compact. (In this case, any subgroup Γ as in Definition 3.1 must be infinite, and cannot be a lattice in G.)

With the help of H. Oh and A. Iozzi, I began a systematic study of the existence of tessellations, starting with the examples of low real rank. Because of the following observation, we assumed $\mathbb{R}\text{-rank}\, G = 2$.

(3.3) **Proposition** (Kulkarni). *If $\mathbb{R}\text{-rank}\, G \le 1$, and neither H nor G/H is compact, then G/H does not have a tessellation.*

3.1. **Homogeneous spaces of** $\mathrm{SL}(3,\mathbb{R})$**.** None of the interesting homogeneous spaces of $\mathrm{SL}(3,\mathbb{R})$ have tessellations. Y. Benoist [Bst, Cor. 1] and G.A. Margulis (unpublished) proved (independently) that $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ does not have a tessellation. Much earlier, T. Kobayashi had shown that the conclusion is true if $\mathbb{R}$ is replaced by either $\mathbb{C}$ or $\mathbb{H}$. Using Benoist's method, H. Oh and I [OW2, Prop. 1.10] generalized this result by replacing $\mathrm{SL}(2,\mathbb{R})$ with any closed, connected subgroup H, such that neither H nor $\mathrm{SL}(3,\mathbb{R})/H$ is compact. (The same argument applies when $\mathbb{R}$ is replaced with either $\mathbb{C}$ or $\mathbb{H}$, so the Kobayashi result also generalizes.) However, the proof of Benoist (which applies in a more general context) relies on a somewhat unpleasant argument to establish one particular lemma. A. Iozzi and I [IW2] provided a shorter and nicer proof, by avoiding any appeal to the lemma. However, we do rely on a consequence of a very nice observation of G.A. Margulis [Mr3] on tempered subgroups.

(3.4) **Theorem** (Benoist, Margulis, Kobayashi, Oh and Witte, Iozzi and Witte). *If*

$$G = \mathrm{SL}(3,\mathbb{R}),\ \mathrm{SL}(3,\mathbb{C}),\ \text{or}\ \mathrm{SL}(3,\mathbb{H}),$$

and neither H nor G/H is compact, then G/H does not have a tessellation.

3.2. **Homogeneous spaces of** $\mathrm{SO}(2,n)$ **and** $\mathrm{SU}(2,n)$**.** All but finitely many of the connected, linear, simple real Lie groups of real rank two are isogenous to either $\mathrm{SO}(2,n)$, $\mathrm{SU}(2,n)$, or $\mathrm{Sp}(2,n)$, so these are the main examples to consider. Here, at last, there are some interesting examples, found by R. Kulkarni [Ku, Thm. 6.1] and T. Kobayashi [Kb1, Prop. 4.9].

(3.5) **Example** (Kulkarni, Kobayashi)**.** From the decompositions

$$\mathrm{SO}(2,2m) = \mathrm{SO}(1,2m)\ \mathrm{SU}(1,m) \qquad \text{and} \qquad \mathrm{SU}(2,2m) = \mathrm{SU}(1,2m)\ \mathrm{Sp}(1,m),$$

it is not difficult to see that each of the following four homogeneous spaces has a tessellation:

- $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$,
- $\mathrm{SO}(2,2m)/\mathrm{SU}(1,m)$,
- $\mathrm{SU}(2,2m)/\mathrm{SU}(1,2m)$, and
- $\mathrm{SU}(2,2m)/\mathrm{Sp}(1,m)$.

When n is even, my work with H. Oh [OW2] and A. Iozzi [IW2] provides a complete description of all the (closed, connected) subgroups H of $\mathrm{SO}(2,n)$, such that $\mathrm{SO}(2,n)/H$ has a tessellation, and a complete description of all the (closed, connected) subgroups H

of $\mathrm{SU}(2,n)$, such that $\mathrm{SU}(2,n)/H$ has a tessellation. However, our classification is not quite complete when n is odd (see 3.10), and we do not yet have any significant results for homogeneous spaces of $\mathrm{Sp}(2,n)$.

The following two theorems state a version of our main results for even n. Our papers also describe the possible subgroups H explicitly, but I omit the details here. The list includes continuous families of subgroups H that were not previously known.

(3.6) **Notation.** For subgroups H_1 and H_2 of G, we write $H_1 \sim H_2$ if there is a compact subset C of G, such that $H_1 \subset C H_2 C$ and $H_2 \subset C H_1 C$.

(3.7) **Notation.** For any connected Lie group H, let

$$d(H) = \dim H - \dim K_H,$$

where K_H is any maximal compact subgroup of H. This is well defined, because all the maximal compact subgroups of H are conjugate.

(3.8) **Theorem** (Oh and Witte)**.** *Assume $G = \mathrm{SO}(2,2m)$, and let H be a closed, connected, subgroup of G, such that neither H nor G/H is compact.*

The homogeneous space G/H has a tessellation if and only if

(1) *$d(H) = 2m$; and*
(2) *either $H \sim \mathrm{SO}(1,2m)$ or $H \sim \mathrm{SU}(1,m)$.*

(3.9) **Theorem** (Iozzi and Witte)**.** *Assume $G = \mathrm{SU}(2,2m)$, and let H be a closed, connected, subgroup of G, such that neither H nor G/H is compact.*

The homogeneous space G/H has a tessellation if and only if

(1) *$d(H) = 4m$; and*
(2) *either $H \sim \mathrm{SU}(1,2m)$ or $H \sim \mathrm{Sp}(1,m)$.*

For each odd n, we have reduced the question to checking only three explicit homogeneous spaces.

(3.10) **Theorem** (Oh and Witte, Iozzi and Witte)**.** *Assume*

$$G = \mathrm{SO}(2,2m+1) \text{ or } \mathrm{SU}(2,2m+1),$$

and that the three homogeneous spaces

$$\mathrm{SO}(2,2m+1)/\,\mathrm{SU}(1,m),\ \mathrm{SU}(2,2m+1)/\,\mathrm{Sp}(1,m) \text{ and } \mathrm{SU}(2,2m+1)/\,\mathrm{SU}(1,2m+1)$$

do not have tessellations.

If H is any closed, connected subgroup of G, such that neither H nor G/H is compact, then G/H does not have a tessellation.

The main results of [OW1] list all the homogeneous spaces of $\mathrm{SO}(2,n)$ that admit a proper action of a noncompact subgroup of $\mathrm{SO}(2,n)$. (This analysis was extended to $\mathrm{SU}(2,n)$ in [IW1].) Our original proof of Theorem 3.8 consisted of a case-by-case analysis to decide whether each of these homogeneous spaces has a tessellation; thus the proof relied on the full strength of [OW1], which required very tedious case-by-case analysis. A. Iozzi and I later observed that the full classification result of [OW1] is not needed; we observed that there is an *a prior* lower bound on $\dim H$ if G/H has a tessellation, and the classification of these subgroups of large dimension can be achieved fairly easily. Therefore, the proofs in [IW2] are relatively painless.

4. Unipotent dynamics

My Ph.D. thesis extended M. Ratner's [Rt1] well-known theorem on rigidity of horocycle flows:

(4.1) **Theorem** (Ratner)**.** *Suppose* Γ *and* Λ *are lattice subgroups of* $G = \mathrm{SL}(2,\mathbb{R})$*, and let* $u = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ *be a nonidentity unipotent element of* G*. If* $\psi \colon G/\Gamma \to G/\Lambda$ *is a* u*-equivariant measure-preserving Borel map, then* ψ *is an affine map (a.e.).*

In short, two translations by unipotent elements are not measure-theoretically isomorphic unless it is obvious from the algebraic setting that this is the case.

4.1. **Isomorphisms.** My thesis generalized Ratner's theorem, by replacing $\mathrm{SL}(2,\mathbb{R})$ with any Lie group G, and allowing u to be any unipotent affine map. I then subsumed this result by considering affine maps that need not be unipotent, but are only assumed to have zero entropy [W3]. (Even before completing my thesis, I had considered some non-unipotent translations [W1].) This zero-entropy assumption is necessary, because no affine map of non-zero entropy can satisfy nearly so strong a rigidity theorem.

Later, Ratner [Rt2] showed that every ergodic probability measure for a unipotent translation has a simple algebraic form. This powerful result, which is related to G. A. Margulis' [Mr1] proof of the Oppenheim Conjecture, has made the earlier work on rigidity of unipotent translations obsolete. I am currently writing an expository article [M2], to make the proof of this important result accessible to a larger audience of geometers, Lie theorists, dynamicists, and number theorists.

4.2. **Measurable quotients.** Using Ratner's measure-classification theorem, I [W5] showed that every quotient of an ergodic unipotent translation has a simple algebraic form; roughly speaking, every quotient is a double-coset space $\Gamma\backslash G/K$. After Ratner [Rt3] extended her theorem to unipotent translations on homogeneous spaces of products of p-adic groups and real groups, I [W10] showed that the classification of quotients can also be extended.

Using a similar idea, C. E. Silva and I [SW] proved that if a totally ergodic flow with quasi-invariant measure has minimal self-joinings, then it has no nontrivial, proper quotients (that is, it is prime). (This generalizes the analogous theorem of D. J. Rudolph and C. E. Silva [RS] on quotients of a single transformation studied in isolation.) More generally, we showed that if a properly ergodic group action with quasi-invariant measure has minimal self-joinings, then every nontrivial quotient is the orbit-space of some closed subgroup of the center of the group.

5. Hamiltonian cycles in Cayley graphs

(5.1) **Definition.** A graph is *vertex-transitive* if its automorphism group acts transitively on the set of vertices. A *Cayley graph* is an especially nice vertex-transitive graph, for which there is a group G of automorphisms that acts simply transitively on the vertices; that is, if x and y are any vertices in the graph, then there is a *unique* $g \in G$ with $g(x) = y$.

It is conjectured that every (connected) Cayley graph has a hamiltonian cycle (that is, a cycle that passes through all the vertices, without repetition), and almost all of my work in graph theory is related to this problem. I wrote a survey [WG] with J. A. Gallian, and it was updated by S J. Curran and J. A. Gallian [CG].

5.1. **Cyclic commutator subgroup of prime-power order.** It is easy to find a hamiltonian cycle if the structure group G of the Cayley graph is abelian, but we are nowhere near a proof of the conjecture in the general case.

It is natural to try to prove that a hamiltonian cycle exists if G is "nearly abelian." Specifically, one might assume that the commutator subgroup of G is "small," in some sense. Perhaps the most general result of this type is my theorem with K. Keating [KW], which allows the commutator subgroup to be cyclic of prime-power order. More recently, E. Dobson, H. Gavlas, J. Morris and I [DGMW] proved a generalization that applies to vertex-transitive graphs, not just Cayley graphs.)

5.2. **Sums of hamiltonian cycles.** If G is abelian, one can find *many* hamiltonian cycles. As a manifestation of this, B. Alspach, S. Locke and I [ALW] showed that the hamiltonian cycles span the space of all cycles. If the number of vertices is odd (and the graph is not the 3×3 torus grid), S. Locke and I [LW1] then extended this work to show that hamiltonian cycles span not only the cycles, which can be thought of as mod-2 flows, but all flows.

J. Morris, D. Moulton and I [MMW] are continuing this work. Assuming G is abelian, the number of vertices is even, and the degree of each vertex is at least 5, we have showed that the span of the hamiltonian cycles is precisely the space of even flows. (This is not the space of all flows unless the graph is bipartite.)

There are some exceptional Cayley graphs of degree 4, such that not every even flow is a sum of hamiltonian cycles, and we hope to be able to determine an exact list of these special graphs.

5.3. ***p*-groups and torus grids.** I [W2] proved there is a hamiltonian cycle in every Cayley graph for which the number of vertices is a prime power. Moreover, a hamiltonian cycle can be found even if each edge of the graph is given a direction (resulting in a *Cayley digraph*), and the cycle is required to traverse each edge in the specified direction.

This general result is perhaps surprising, because there are many Cayley digraphs—even ones with abelian structure group—that have no hamiltonian cycle. For example, an $m \times n$ torus grid, with the natural directions on its edges, (also known as the cartesian product of two directed cycles) is a Cayley digraph with structure group $\mathbb{Z}_m \times \mathbb{Z}_n$, and has no hamiltonian cycle if $\gcd(m,n) = 1$ [HS, Prop. 5.6]. L. Penn and I [PW] determined the lengths of all the cycles in any torus grid. S. Curran and I [CW] proved that the analogous 3-dimensional torus grids (and higher-dimensional grids) all have hamiltonian cycles.

For 3-dimensional torus grids in which all of the sides have the same length (that is, for the natural Cayley digraph on the abelian group $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_m$), D. Austin, H. Gavlas and I [AGW] determined exactly which pairs of vertices can be joined by a hamiltonian path. Namely, if $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, and there is a hamiltonian path from v to w, then it is easy to see that

$$v_1 + v_2 + v_3 \equiv w_1 + w_2 + w_3 + 1 \pmod{m}.$$

We proved the converse.

5.4. **Circulant digraphs.** Circulant graphs are a special case of Cayley graphs on abelian groups, so every connected, circulant graph has a hamiltonian cycle. The situation is different in the directed case: some connected, circulant digraphs are not hamiltonian.

For those of outdegree two, R. A. Rankin found a simple arithmetic criterion that determines which are hamiltonian.

In contrast, little is known about the hamiltonicity of circulant digraphs of outdegree three (or more). I [WG, p. 301] found one nonhamiltonian example many years ago (see Fig. 6 on p. 9). S. Locke and I [LW2] recently showed that there are infinitely many nonhamiltonian examples of outdegree three.

5.5. **Transitive groups of prime-squared degree.** It is well known that a classification of all transitive groups of prime degree follows from the Classification of Finite Simple Groups. As a sequel to this, E. Dobson and I [DW] began the classification of all transitive groups of prime-squared degree.

One of our main results determines all of the transitive permutation groups of degree p^2 whose order is not divisible by p^{p+1}.

In the case where $|G|$ is divisible by p^{p+1}, we have a complete classification for most primes p; specifically, our classification is complete unless either $p \in \{11, 23\}$ or $p = (q^d - 1)/(q-1)$, for some prime-power q and natural number d.

These are not results about hamiltonian cycles, but (as our paper explains) they do have applications to other questions about Cayley graphs.

6. Miscellaneous work

6.1. **$\mathbb{Q}$-forms of real representations.** It is easy to see, from the theory of highest weights, that if $\mathfrak{g}$ is an $\mathbb{R}$-split, semisimple Lie algebra over $\mathbb{R}$, then every $\mathbb{C}$-representation of $\mathfrak{g}$ has an $\mathbb{R}$-form. (That is, if $V_{\mathbb{C}}$ is a representation of $\mathfrak{g}$ over $\mathbb{C}$, then there is a real representation V of $\mathfrak{g}$, such that $V_{\mathbb{C}} \cong V \otimes_{\mathbb{R}} \mathbb{C}$.) Because every semisimple Lie algebra over $\mathbb{C}$ has an $\mathbb{R}$-split real form, this leads to the following immediate conclusion:

(6.1) *Remark.* Any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has a real form $\mathfrak{g}$, such that every $\mathbb{C}$-representation of $\mathfrak{g}$ has a real form.

I [M1] proved the analogous statement with the field extension $\mathbb{C}/\mathbb{R}$ replaced with $\mathbb{R}/\mathbb{Q}$.

(6.2) **Theorem** (Morris)**.** *Any real semisimple Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}}$, such that every real representation of $\mathfrak{g}_{\mathbb{Q}}$ has a $\mathbb{Q}$-form.*

In the special case where $\mathfrak{g}$ is compact, the theorem was proved by M. S. Raghunathan [Rg, §3]. This special case was independently rediscovered by P. Eberlein [Eb], and a very nice proof was found by R. Pink and G. Prasad (personal communication). When $\mathfrak{g}$ is compact, these authors showed that the "obvious" $\mathbb{Q}$-form of $\mathfrak{g}$ has the desired property.

At the other extreme, where $\mathfrak{g}$ is $\mathbb{R}$-split, we may take $\mathfrak{g}_{\mathbb{Q}}$ to be any $\mathbb{Q}$-split $\mathbb{Q}$-form of $\mathfrak{g}$.

The general case is a combination of the two extremes, and the desired $\mathbb{Q}$-form can be obtained from a Chevalley basis of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by slightly modifying a construction of A. Borel [Bo]. I gave two different proofs that this $\mathbb{Q}$-form has the desired property: one proof is by the method of Pink and Prasad, using a little bit of number theory, and the other proof is by reducing to the compact case, so Raghunathan's theorem applies.

It would be interesting to characterize the semisimple Lie algebras $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$, such that every real representation has a $\mathbb{Q}$-form. For example, work of J. Tits [Ti] implies that every $\mathbb{Q}$-form of the compact group $\mathrm{Sp}(n)$ has this property. On the other hand, it is important to note that there exist examples of $\mathbb{Q}[i]$-split Lie algebras that do not have this property. (Real representations of such a Lie algebra can be realized over both $\mathbb{Q}[i]$ and $\mathbb{R}$, but not over $\mathbb{Q}[i] \cap \mathbb{R} = \mathbb{Q}$.)

6.2. **Homogeneous Lorentz manifolds.** A. Zeghib [Ze] classified the compact homogeneous spaces that admit an invariant Lorentz metric. I [W13] removed the assumption of compactness, but added the restriction that the transitive group G is almost simple. My result was used in work of S. Adams [Ad] on nontame actions on Lorentz manifolds.

My starting point was a special case of a theorem of N. Kowalsky [Kw3, Thm. 5.1].

(6.3) **Theorem** (Kowalsky). *Let G/H be a nontrivial homogeneous space of a connected, almost simple Lie group G with finite center. If there is a G-invariant Lorentz metric on G/H, then either*

(1) *there is also a G-invariant Riemannian metric on G/H; or*
(2) *G is locally isomorphic to $\mathrm{SO}(k,n)$, for some $k \in \{1,2\}$ and some $n \geq k+1$.*

It is easy to characterize the homogeneous spaces that arise in Conclusion (1) Kowalsky's theorem. Thus, I will discuss only the cases that arise in Conclusion (2). It is well known [Kw2, Egs. 2 and 3] that

$$\mathrm{SO}(1,n)^\circ / \mathrm{SO}(1,n-1)^\circ \text{ and } \mathrm{SO}(2,n)^\circ / \mathrm{SO}(1,n)^\circ \text{ have invariant Lorentz metrics.}$$

Also, for any discrete subgroup Γ of $\mathrm{SO}(1,2)$, the Killing form provides

$$\text{an invariant Lorentz metric on } \mathrm{SO}(1,2)^\circ/\Gamma.$$

I showed that these are the only examples (up to local isomorphism, finite covers, and passing to conjugate subgroups).

N. Kowalsky [Kw2, Thm. 4] announced a much more general result than my result for $\mathrm{SO}(2,n)$, but it seems that she did not publish a proof before her premature death. She [Kw2, Thm. 3] also announced a version of my result for $\mathrm{SO}(1,n)$ (with much more general hypotheses and a somewhat weaker conclusion), and a proof appears in her Ph.D. thesis [Kw1, Cor. 6.2].

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