Classical Analytic Theory of L-functions

Lecture 4: Applications

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1 The Prime Number Theorem

Theorem 1 (**Wiener-Ikehara Tauberian Theorem**) *Let* $f(s) = \sum_{n=1}^{\infty} a_n/n^s$*, with* $a_n \geq 0$*, and* $g(s) = \sum_{n=1}^{\infty} b_n/n^s$ *be two Dirichlet series with* $|b_n| \leq a_n$ *for all* n*. Assume that* $f(s)$ *and* $g(s)$ *extend analytically to* $\Re(s) \geq 1$ *except possibly at* $s = 1$ *where they have simple poles with residues* R *and* r *(which may be zero) respectively. Then*

$$\sum_{n \leq x} b_n \sim rx,$$

as $x \to \infty$*.*

Proposition 2 *Let* f *be a normalized eigenform of weight* k *and level* N*. Then*

$$\sum_{n \leq x} |a_f(n)|^2 \sim rx,$$

where

$$r = \frac{12(4\pi)^{k-1}}{N(k-1)! \prod_{p|N}(1+\frac{1}{p})} \langle f, f \rangle,$$

and

$$\sum_{n=1}^{\infty} b_f(n) \sim \frac{\pi^2}{6} \prod_{p|N} (1 - \frac{1}{p^2}) rx,$$

where

$$b_f(n) = \sum_{\substack{n=d^2 m \\ (d,N)=1}} |a_f(m)|^2.$$

Proof This is a direct corollary of the Tauberian Theorem and calculation of the residues of $L(f \times f, s)$ and $L(f \otimes f, s)$ at $s = 1$. ☐

Proposition 3 *Let f be a normalized eigenform of level N and weight k. Then*

$$\sum_{p \le x} \lambda_f(p) \log p = o(x).$$

Proof With notation of Lecture 2, we have

$$-\frac{L'}{L}(f, s) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s},$$

where $\Lambda_f(n) = (\alpha_1(p)^k + \alpha_2(p)^k) \log p$ if $n = p^k$, and $\Lambda_f(n) = 0$ otherwise. Since Ramanujan-Petersson conjecture is true in this case, we have

$$|\Lambda_f(n)| \le 2 \log n.$$

By non-vanishing result of Lecture 2, $-\frac{L'}{L}(f, s)$ is analytic at $s = 1$. Also $-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\log n}{n}$ has an analytic continuation to the whole complex plane with an exception of a simple pole at $s = 1$. So by the Tauberian Theorem

$$\sum_{n \le x} \Lambda_f(n) = \sum_{p \le x} \lambda_f(p) \log p + \sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \Lambda_f(p^\alpha) = o(x). \tag{1}$$

Now let

$$\theta(x) = \sum_{p \le x} \log p.$$

(This should not be confused with the theta function.) Then we have

$$|\sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \Lambda_f(p^\alpha)| \le 2 \sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \log p = \theta(x^{1/2}) + \cdots + \theta(x^{1/r}),$$

where r is as large as $\log x$. Since $\theta(x) \le x \log x$, we conclude that

$$|\sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \Lambda_f(p^\alpha)| \ll \sqrt{x} \log^2 x.$$

Now applying this bound in (1) implies the result. ☐

2 The Prime Number Theorem With the Remainder

Complex Analysis 4 (**Perron's formula**) *Let* $x > 0$, $a > 0$, $T > 0$. *Let* $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a Dirichlet series absolutely convergent in* $\Re(s) > a - \epsilon$. *Then if* x *is a non-integer*

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^a}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^a |\log \frac{x}{n}|} \right).$$

Exercise 5 *Show that*

$$\sum_{n=1}^{\infty} \frac{\log n}{n^a |\log \frac{x}{n}|} = O\left(\frac{1}{(a-1)^2} + x^{1-a} \log^2 x \right),$$

where $1 < a \leq 2$ *and* x *is a half-integer* (*i.e.* $x = \frac{2k+1}{2}$ *for an integer* k). *The implied constant is absolute.*

Exercise 6 *For* $t > 1$ *and* $1 - \frac{c}{\log(Nk(|t|+3))} \leq \sigma \leq 1 + \frac{c}{\log(Nk(|t|+3))}$, *we have*

$$\frac{L'}{L}(f, \sigma + it) \ll \log(Nk(|t|+3)).$$

Here c *is the constant coming from the almost zero-free region.*

Theorem 7 (**Moreno**) *Let* f *be a normalized newform of level* N *and weight* k. *Then there exists an absolute constant* $c > 0$ *such that* $L(f,s)$ *has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(Nk(|t|+3))}$$

except possibly one simple real zero $\beta < 1$. *Moreover,*

$$\sum_{p \leq x} \lambda_f(p) \log p = -\frac{x^\beta}{\beta} + O(\sqrt{Nk}\ x \exp(-c_1 \sqrt{\log x}))$$

for $x \geq 2$, *where* $c_1 > 0$ *and the implied constant is absolute.*

Proof Since f is a normalized newform $L(f,s)$ is self-dual. So the first part of the theorem is a simple corollary of almost zero-free region theorem in Lecture 2.
For the proof of the asymptotic formula (upper bound), let x be a half-integer, $T \geq 3$, and $a = 1 + \frac{c}{\log(NkT)}$. So by Perron's formula, the bound $|\Lambda_f(n)| \leq 2 \log n$ and Exercise 5, we have

$$\sum_{n \leq x} \Lambda_f(n) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{L'}{L}(f,s) \frac{x^s}{s} ds + O\left(\frac{x^a}{T} (\frac{1}{(a-1)^2} + x^{1-a} \log^2 x) \right).$$

Next let $b = 1 - \dfrac{c}{\log(NkT)}$, where c is the constant coming from the first part of theorem. We consider a rectangle $R_T = R_1 \cup (-R_2) \cup (-R_3) \cup R_4$, where

$$\begin{array}{rcll} R_1 & : & s = a + it, & -T \le t \le T, \\ R_2 & : & s = \sigma + iT, & b \le \sigma \le a, \\ R_3 & : & s = b + it, & -T \le t \le T, \\ R_4 & : & s = \sigma + iT, & b \le \sigma \le a. \end{array}$$

Since $\frac{L'}{L}(f,s)$ has a simple pole of residue 1 at $s = \beta$ (in case the exceptional zero exists), we have

$$\sum_{n \le x} \Lambda_f(n) = -\frac{x^\beta}{\beta} - \frac{1}{2\pi i}\left(\int_{-R_2} + \int_{-R_3} + \int_{R_4}\right) + O\left(\frac{x^a}{T}\left(\frac{1}{(a-1)^2} + x^{1-a}\log^2 x\right)\right). \qquad (2)$$

Now by employing Exercise 6 we have

$$\frac{1}{2\pi i}\int_{-R_2} -\frac{L'}{L}(f,s)\frac{x^s}{s}ds \ll \frac{x^a}{T}.$$

A similar bound holds for $\frac{1}{2\pi i}\int_{R_4}$. Also

$$\frac{1}{2\pi i}\int_{-R_3} -\frac{L'}{L}(f,s)\frac{x^s}{s}ds \ll x^b \log(NkT)\log T.$$

Applying these bounds in (2) yields

$$\sum_{n \le x} \Lambda_f(n) = -\frac{x^\beta}{\beta} + O\left(x^b \log(NkT)\log T + \frac{x^a}{T} + \frac{x^a}{T(a-1)^2} + \frac{x\log^2 x}{T}\right).$$

Now let $T = x^{a-b} = x^{\frac{2c}{\log(NkT)}}$. We have

$$\begin{aligned} \sum_{n \le x} \Lambda_f(n) &= -\frac{x^\beta}{\beta} + O(x^b \log(NkT)\log T + x^b \log^2(NkT) + x^{1-a}x^b\log^2 x) \\ &= -\frac{x^\beta}{\beta} + O(x^b \log^2(NkT)\log^2 T) \\ &= -\frac{x^\beta}{\beta} + O(\sqrt{Nk}\ x\exp(-c_1\sqrt{\log x})), \end{aligned}$$

for some $c_1 > 0$. This implies that

$$\sum_{p \le x} a_f(p)\log p = -\frac{x^\beta}{\beta} + O(\sqrt{Nk}x\exp(-c_1\sqrt{\log x})).$$

$\square$

Note The asymptotic formula in the previous theorem in fact is an upper bound. Alternately we can write the formula as

$$\sum_{p\le x} a_f(p)\log p = O(\sqrt{N}kx\exp(-c_1\sqrt{x})),$$

where the implied constant depends on β. Since the position of the exceptional zero is not clear, the implied constant is not effectively computable.

Note It is proved by Hoffstein and Ramakrishnan that there is no exceptional zero for normalized newforms. So we can drop the term $-\frac{x^\beta}{\beta}$ in Moreno's theorem.

3 An Effective Lower Bound

Exercise 8 *Show that*

$$\frac{1}{2\pi i}\int_{(2)}\frac{x^s}{s(s+1)\cdots(s+r)}ds = \begin{cases} \frac{1}{r!}(1-\frac{1}{x})^r, & x>1;\\ 0, & 0<x\le 1.\end{cases}$$

Proposition 9 (**Hoffstein and Lockhart**) *Let $L(f,s)\in\mathcal{IK}$ be an L-function with positive coefficients and a single simple pole at $s=1$ of residue r. Suppose that $L(f,s)$ satisfies a growth condition on the line $\Re(s)=1/2$ of the form*

$$|L(f,\frac{1}{2}+it)|\le M(|t|+3)^B$$

for some constant B. If $L(f,s)$ has no real zeros in the range

$$1-\frac{1}{\log M}<\sigma<1,$$

then there exists an effective constant $c=c(B)>0$ such that

$$\frac{1}{r}\le c\log M.$$

Proof Let $\frac{1}{2}<\beta<1$ and let r be a fixed integer greater than B. (We should not confuse β with the exceptional zero.) Using Exercise 8 and the absolute convergence of $L(f,s+\beta)$ in the range of integration, we get

$$\frac{1}{2\pi i}\int_{(2)}\frac{L(f,s+\beta)x^s}{s(s+1)\cdots(s+r)}ds = \frac{1}{r!}\sum_{n<x}\frac{\lambda_f(n)}{n^\beta}(1-\frac{n}{x})^r.$$

Since $\lambda_f(n)$ are non-negative, and $\lambda_f(1) = 1$, we have for all $x \geq 2$,

$$1 \ll \frac{1}{2\pi i} \int_{(2)} \frac{L(f, s+\beta)x^s}{s(s+1)\cdots(s+r)} ds. \tag{3}$$

From the growth condition on the line $\Re(s) = \frac{1}{2}$ and the Phragmen-Lindelöf principle, we have

$$L(f, \sigma + it) = O(|t|^B)$$

for all $\frac{1}{2} \leq \sigma \leq 3$ and $t \geq 1$. Thus we can shift the line of integration to $\Re(s) = \frac{1}{2} - \beta < 0$, picking up residues at $s = 0, 1-\beta$. Using the bound on the line $\Re(s) = \frac{1}{2}$, the right-hand side of (3) becomes

$$\frac{rx^{1-\beta}}{(1-\beta)(2-\beta)\cdots(r+1-\beta)} + \frac{L(f,\beta)}{r!} + O(Mx^{\frac{1}{2}-\beta}).$$

Taking $x = M^C$, for C a sufficiently large constant, we get

$$1 \ll \frac{rM^{C(1-\beta)}}{1-\beta} + L(f, \beta). \tag{4}$$

Now we choose

$$\beta = 1 - \frac{1}{\log M}.$$

Since $L(f, s)$ has a simple pole at $s = 1$, and is positive for $\sigma > 1$, we must have $L(f, \beta) \leq 0$. Then (4) yields

$$\frac{1}{r} \ll \log M$$

as desired. ☐

Note If $L(f, s) \in \mathcal{IK}$ the growth condition on the line $\Re(s) = \frac{1}{2}$ will be satisfied if we choose M as a suitable power of the conductor of $L(f, s)$ and the product of its local factor at ∞.

Definition 10 *The symmetric square L-function associated to a normalized eigenform f of weight k and level N is defined as*

$$L(\mathrm{sym}^2 f, s) = \frac{L(f \otimes f, s)}{\zeta_N(s)}.$$

From Rankin-Selberg theory it is clear that $L(\mathrm{sym}^2 f, s)$ has a meromorphic continuation to $\mathbb{C}$. In 1975 Shimura proved that the symmetric square L-function in fact has an analytic continuation to the whole complex plane.

For square-free N and newform f, the symmetric square L-function associated to f satisfies a functional equation. Let

$$L_\infty(\mathrm{sym}^2, s) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right),$$

and let

$$\Lambda(\mathrm{sym}^2 f, s) = N^s L_\infty(\mathrm{sym}^2, s)L(\mathrm{sym}^2, s).$$

Then the symmetric square L-function satisfies

$$\Lambda(\mathrm{sym}^2 f, s) = \Lambda(\mathrm{sym}^2 f, 1-s).$$

Exercise 11 *Show that $L(\mathrm{sym}^2 f, s)$ has an Euler product on $\Re(s) > 1$.*

Theorem 12 (**Goldfeld, Hoffstein and Lieman**) *Let f be a normalized newform of square-free level N and weight k. Then there exists an absolute constant $c > 0$ such that $L(\mathrm{sym}^2 f, s)$ has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(kN)}.$$

Proof Consider the L-function

$$L(g,s) = \zeta(s)L(\mathrm{sym}^2 f, s)^2 L(\mathrm{sym}^2 f \otimes \mathrm{sym}^2 f, s) = \zeta(s)L(\mathrm{sym}^2 f, s)^3 \frac{L(\mathrm{sym}^2 f \otimes \mathrm{sym}^2 f, s)}{L(\mathrm{sym}^2 f, s)}.$$

The last L-function is a special case of the symmetric square of a cusp form on $GL(3)$ and has been shown by Bump and Ginzburg to have a simple pole at $s = 1$. Hence $L(g,s)$ has a pole of order 2 at $s = 1$, whereas any real zero of $L(\mathrm{sym}^2 f, s)$ is a zero of $L(g,s)$ of order ≥ 3. By local computations one checks that $\Lambda_g(n) \geq 0$ for $(n, q(g)) = 1$, hence the result follows from Goldfeld, Hoffstein and Lieman's Lemma in Lecture 2. $\square$

Exercise 13 *For $\Re(s) > 0$ define*

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

(*i*) *Show that $f(s) = (1 - \frac{1}{2^{s-1}})\zeta(s)$ for $\Re(s) > 1$.*

(*ii*) *From part* (*i*) *deduce a meromorphic continuation of $\zeta(s)$ into the half-plane $\Re(s) > 0$.*

(*iii*) *Show that $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.*

(*iv*) *From here conclude that $L(f \otimes f, \sigma) \neq 0$, in the region given in the previous theorem.*

Corollary 14 (**Hoffstein and Lockhart**) *Let f be a normalized newform of square-free level N and weight k. Then there exists an effective constant c (depends only on k) such that*

$$\langle f, f\rangle = \|f\|^2 \geq c\frac{N}{\log N}.$$

Proof This is immediate from the previous exercise and Hoffstein and Lockhart's proposition. Note that the residue of $L(f \otimes f, s)$ at $s = 1$ is

$$\frac{\phi(N)\pi(4\pi)^k}{2N^2(k-1)!}\langle f, f\rangle.$$

□

Exercise 15 *Let f be a newform of square-free level N and weight k with Petersson norm $\|f\| = 1$. Let $\rho(1)$ be the first Fourier coefficient of f. Then prove that there exists an effective constant c (depends only on k) such that*

$$|\rho(1)|^2 \leq c\frac{\log N}{N}.$$

4 Bounds for the Fourier Coefficients of Cusp forms

Theorem 16 (**Landau [also Chandrasekharan and Narasimhan]**) *Let $L(f, s) = \sum_{n=1}^{\infty}$ be a Dirichlet series with non-negative coefficients $\lambda_f(n)$ and converging for $\Re(s)$ sufficiently large. Assume that $L(f, s)$ has a meromorphic continuation to $\mathbb{C}$ with at most poles of finite order at $s = 0$, 1; assume also that $L(f, s)$ is of finite order in the half plane $\Re(s) \geq -1$, and it satisfies a functional equation of the form*

$$q(f)^s\gamma(f, s)L(f, s) = \epsilon(f)q(f)^{1-s}\gamma(f, 1-s)L(f, 1-s)$$

for some constant $\epsilon(f), q(f) > 0$, where

$$\gamma(f, s) = \prod_{i=1}^{d}\Gamma(\alpha_i s + \beta_i),$$

for some $d \geq 1$ and $\alpha_i \geq 0$, $\beta_i \in \mathbb{C}$ for $1 \leq i \leq d$. Setting $\eta = \sum_{i=1}^{d}\alpha_i$, one has

$$\sum_{n \leq x}\lambda(n) = P_{r-1}(\log x)x + O(x^{\frac{2\eta-1}{2\eta+1}}\log^{r-1} n),$$

where r is the order of pole of $L(f, s)$ at $s = 1$ and P_{r-1} is a polynomial of degree $r - 1$ that depends only on $L(f, s)$. The implied constant also depends only on $L(f, s)$.

Theorem 17 (**Rankin**) *Let f be a cusp form of weight k and level* 1*. Then*

$$\sum_{n\leq x}|a_f(n)|^2 = c_f x + O(x^{\frac{3}{5}}),$$

where

$$c_f = \frac{12(4\pi)^{k-1}}{(k-1)!}\langle f,f\rangle.$$

Proof Let $b_f(n)$ denote the coefficients of $L(f\otimes f,s)$. Since

$$L(f\times f,s) = \frac{1}{\zeta(2s)}L(f\otimes f,s),$$

we have

$$|a_f(n)|^2 = \sum_{n=d^2m}\mu(d)b_f(m), \tag{5}$$

$\mu(d)$ denotes the Möbius function. Now one can check that $L(f\otimes f,s)$ satisfies the conditions of Landau's Theorem and so

$$\sum_{n\leq x}b_f(n) = \frac{\pi^2}{6}c_f x + O(x^{\frac{3}{5}}).$$

Now from Proposition 2, we have

$$c_f = \frac{12(4\pi)^{k-1}}{(k-1)!}\langle f,f\rangle.$$

Next from (5) and the above asymptotic formula we have

$$\begin{aligned}
\sum_{n\leq x}|a_f(n)|^2 &= \sum_{n\leq x}\sum_{n=d^2m}\mu(d)b_f(m)\\
&= \sum_{d^2m\leq x}\mu(d)b_f(m)\\
&= \sum_{m\leq\frac{x}{d^2}}b_f(m)\sum_{d\leq\sqrt{x}}\mu(d)\\
&= \sum_{d\leq\sqrt{x}}\mu(d)(\frac{\pi^2}{6}c_f\frac{x}{d^2} + O\left((\frac{x}{d^2})^{\frac{3}{5}}\right)\\
&= c_f x\frac{\pi^2}{6}\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2} + O(x^{\frac{3}{5}}\sum_{d\leq\sqrt{x}}\frac{1}{d^{\frac{6}{5}}})\\
&= c_f x + O(x^{\frac{3}{5}}).
\end{aligned}$$

□

Corollary 18 (**Rankin**) *$a_f(n) = O_f(n^{\frac{3}{10}})$ and $\tau(n) = O(n^{\frac{29}{5}})$.*

Exercise 19 *Let $L(f,s) \in \mathcal{IK}$ be an L-function of degree d. Also assume that $L(f \otimes f, s)$ exists and $\lambda_{f\otimes f}(n) \geq 0$ for $(n, q(f)) \neq 1$. Then show that for the local parameters $\alpha_i(p)$ for unramified prime p we have*

$$|\alpha_i(p)| \leq p^{\frac{1}{2} - \frac{1}{d^2+1}}.$$