Classical Analytic Theory of L-functions

Lecture 3: Poles of L-functions, Examples

Part B: Rankin-Selberg Convolution

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Let $z = x + iy$ be a point in the upper half-plane $\mathbb{H}$, and let $s = \sigma + it$ be a point in the complex plane $\mathbb{C}$. Let

$$f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n) e^{2\pi i n z}$$

and

$$g(z) = \sum_{n=1}^{\infty} \hat{a}_g(n) e^{2\pi i n z}$$

be cusp forms of weight k and level N. We set

$$\delta(f, g) = y^{k-2} f(z) \overline{g(z)}.$$

Recall that for $Re(s) > 1$, the L-functions attached to f and g are defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

and

$$L(g, s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s}$$

where

$$a_f(n) = \frac{\hat{a}_f(n)}{n^{\frac{k-1}{2}}}, \ a_g(n) = \frac{\hat{a}_g(n)}{n^{\frac{k-1}{2}}}$$

for $n = 1, 2, 3, \cdots$.

Definition 1 *The Rankin-Selberg convolution of $L(f,s)$ and $L(g,s)$ is defined by*

$$L(f\times g,s)=\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}.$$

The modified Rankin-Selberg convolution of $L(f,s)$ and $L(g,s)$ is defined by

$$L(f\otimes g,s)=\zeta_N(2s)L(f\times g,s)=\zeta_N(2s)\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s}$$

where $\zeta_N(s)=\sum_{\substack{n=1\\ \text{g.c.d.}(n,N)=1}}^{\infty}\frac{1}{n^s}=\prod_{p\nmid N}\left(1-\frac{1}{p^s}\right)^{-1}$ *is the Riemann zeta-function with the Euler p-factors corresponding to $p\mid N$ removed.*

The main goal of this section is to study the analytic properties of $L(f\times g,s)$. We will see that the analytic continuation and the functional equation of the Epstein zeta-function $E(z,s)$ will result in the analytic continuation and the functional equation for the Rankin-Selberg convolution $L(f\times g,s)$.

In Lemma 3 we will relate the Rankin-Selberg convolution $L(f\times g,s)$ to a double integral on a certain region of the upper half-plane. To do this we need the following lemma.

Lemma 2 *For any fixed $y>0$,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}f(z)\overline{g(z)}dx=\sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}.$$

Proof We have

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}f(z)\overline{g(z)}dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\hat{a}_f(m)e^{2\pi im(x+iy)}\overline{\sum_{n=1}^{\infty}\hat{a}_g(n)e^{2\pi in(x+iy)}}\right)dx\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\hat{a}_f(m)\overline{\hat{a}_g(n)}e^{2\pi i(m-n)x}e^{-2\pi(m+n)y}\right)dx.\end{aligned}$$

Interchanging the order of summation and integration yields

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}f(z)\overline{g(z)}dx &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\hat{a}_f(m)\overline{\hat{a}_g(n)}e^{-2\pi(m+n)y}\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi i(m-n)x}dx\right)\\ &= \sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}.\end{aligned}$$

The proof is complete. □

Lemma 3 *For Re(s) > 1 we have the following integral representation for the Rankin-Selberg convolution* $L(f \times g, s)$

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \iint_S y^{s+k-2} f(z)\overline{g(z)}dxdy \\ &= \iint_S y^s \delta(f,g) dxdy\end{aligned}$$

where S is the strip $|x| \leq \frac{1}{2}$ *and* $y > 0$.

Proof We have

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= (4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{n=1}^{\infty}\frac{a_f(n)\overline{a_g(n)}}{n^s} \\ &= \sum_{n=1}^{\infty}\left\{\frac{\hat{a}_f(n)\overline{\hat{a}_g(n)}}{n^{k-1}}\frac{(4\pi)^{-s-k+1}}{n^s}\Gamma(s+k-1)\right\} \\ &= \sum_{n=1}^{\infty}\left\{\hat{a}_f(n)\overline{\hat{a}_g(n)}(4\pi n)^{-s-k+1}\Gamma(s+k-1)\right\}.\end{aligned}$$

Note that by the change of variable $t \mapsto 4\pi ny$, $\Gamma(s+k-1)$ can be written as

$$\Gamma(s+k-1) = (4\pi n)^{s+k-1}\int_0^{\infty} e^{-4\pi ny} y^{s+k-2} dy.$$

So

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \sum_{n=1}^{\infty}\left\{\hat{a}_f(n)\overline{\hat{a}_g(n)}\int_0^{\infty} e^{-4\pi ny}y^{s+k-2}dy\right\} \\ &= \int_0^{\infty} y^{s+k-2}\left\{\sum_{n=1}^{\infty}\hat{a}_f(n)\overline{\hat{a}_g(n)}e^{-4\pi ny}\right\}dy.\end{aligned}$$

Now by applying Lemma 2 we get

$$\begin{aligned}(4\pi)^{-s-k+1}\Gamma(s+k-1)L(f\times g,s) &= \int_0^{\infty} y^{s+k-2}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(z)\overline{g(z)}dx\right\}dy \\ &= \iint_S y^{s+k-2} f(z)\overline{g(z)}dxdy \\ &= \iint_S y^s\delta(f,g)dxdy.\end{aligned}$$

This completes the proof. □

Our next step is to rewrite the double integral in the statement of the previous lemma as a new integral on a fundamental domain for $\Gamma_0(N)$.

Lemma 4 *We have*

$$\iint_S y^s \delta(f,g)dxdy = \iint_{D_0(N)} y^s \delta(f,g) F_N(z,s) dxdy$$

where

$$F_N(z,s) = 1 + \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}}$$

and $D_0(N)$ is a fundamental domain for $\Gamma_0(N)$.

Proof Let

$$\Gamma_\infty = \{\gamma \in \Gamma : \ \gamma\infty = \infty\} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : \ b \in \mathbb{Z} \right\}.$$

Γ_∞ is a subgroup of Γ and it is clear that the strip $S = \{(x,y) : \ |x| \leq \frac{1}{2}, \ y > 0\}$ is a fundamental domain for Γ_∞. For any two matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ in $GL_2(\mathbb{Z})$, the right cosets $\Gamma_\infty\gamma$ and $\Gamma_\infty\gamma'$ are equal if and only if $(c,d) = \pm(c',d')$. So the right cosets of Γ_∞ in $\Gamma_0(N)$ are in one to one correspondence with the pairs (c,d) where $c \geq 0$. Therefore we can choose a set of representative $\mathcal{T}$ for the right cosets of Γ_∞ in $\Gamma_0(N)$ as follows:

$$\mathcal{T} = \{(0,1)\} \cup \{(c,d) : \ c > 0, \ N|c, \ (c,d) = 1\}.$$

We claim that for any pair (c,d) in $\mathcal{T}$, there is a unique transformation

$$\gamma_{c,d} : z_1 \to z = \frac{az_1 + b}{cz_1 + d}$$

that maps $D_0(N)$ into S. This is true for the pair $(0,1)$. For other pairs in $\mathcal{T}$, note that since $\infty \in D_0(N)$,

$$\left|\frac{a}{c}\right| = |\gamma_{c,d}(\infty)| \leq \frac{1}{2}.$$

Since $ad - bc = 1$, equality holds only if $c = 2$, $a = \pm 1$. We consider two cases.
If $c \neq 2$, then there is exactly one solution in a, b of the equation $ad - bc = 1$ for which $\left|\frac{a}{c}\right| < \frac{1}{2}$. Since $\gamma_{c,d}D_0(N)$ has the unique cusp $\frac{a}{c}$ in S, and this cusp is not on either of the lines $|x| = \frac{1}{2}$, the whole of $\gamma_{c,d}D_0(N)$ lies in S.

If $c = 2$, then $a = \pm 1$. Suppose that, for example, $\gamma_{c,d}$ takes ∞ to the cusp $-\frac{1}{2}$ and takes $D_0(N)$ into S. Then the transformation $\gamma_{c,d}(z_1) + 1$ has the same c, d and maps $D_0(N)$ outside S (touching the line $x = \frac{1}{2}$), and therefore corresponds to the other solution. Hence exactly one of the transformations $\gamma_{c,d}(z_1)$ or $\gamma_{c,d}(z_1) + 1$ has the desired property. The claim is proved.

This shows that the strip S can be written as the disjoint union of $\gamma_{c,d} D_0(N)$'s

$$S = \bigcup_{(c,d)\in\mathcal{T}} \gamma_{c,d} D_0(N).$$

Therefore, we have

$$\iint_S y^s \delta(f,g) dxdy = \sum_{(c,d)\in\mathcal{T}} \iint_{\gamma_{c,d}D_0(N)} y^s \delta(f,g) dxdy.$$

Now let $z_1 = x_1 + iy_1$. Changing variable $z_1 \mapsto z = \dfrac{az_1 + b}{cz_1 + d}$ yields

$$\begin{aligned} \iint_S y^s \delta(f,g) dxdy &= \sum_{(c,d)\in\mathcal{T}} \iint_{D_0(N)} \left(\frac{y_1}{|cz_1 + d|^2} \right)^s \delta(f,g) dx_1 dy_1 \\ &= \iint_{D_0(N)} \left(y_1^s \delta(f,g) \sum_{(c,d)\in\mathcal{T}} \frac{1}{|cz_1 + d|^{2s}} \right) dx_1 dy_1. \end{aligned}$$

By considering the definition of $\mathcal{T}$ in the last integral, we have

$$\begin{aligned} \iint_S y^s \delta(f,g) dxdy &= \iint_{D_0(N)} y^s \delta(f,g) \left\{ 1 + \sum_{\substack{c=1 \\ N|c}}^{\infty} \sum_{\substack{d=-\infty \\ \text{g.c.d.}(d,c)=1}}^{\infty} \frac{1}{|cz + d|^{2s}} \right\} dxdy \\ &= \iint_{D_0(N)} y^s \delta(f,g) F_N(z,s) dxdy. \end{aligned}$$

The proof is complete. $\square$

Now we will show that $F_N(z,s)$ has a representation in terms of the Epstein zeta-function. First we recall the definition of the Möbius function.

The *Möbius function* $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r,\ p_i \neq p_j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5 *We have*

$$2\zeta_N(2s)F_N(z,s) = \sum_{d|N} \frac{\mu(d)}{d^{2s}} E(\frac{N}{d}z, s).$$

Proof The idea is to evaluate the double sum

$$S = \sum_{\substack{m,n \\ (n,N)=1}}{}' \frac{1}{|mNz+n|^{2s}}$$

in two different ways.

On one hand we have

$$\begin{aligned}
S &= 2 \sum_{\substack{n=1 \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{n^{2s}} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,N)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\
&= 2\zeta_N(2s) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{|mNz+n|^{2s}} \\
&= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,m)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}}.
\end{aligned}$$

Note that since g.c.d.$(n,N)=1$, then g.c.d.$(n,m)=$ g.c.d.(n,mN). So

$$\begin{aligned}
S &= 2\zeta_N(2s) + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=k}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\
&= 2\zeta_N(2s) + 2\zeta_N(2s)\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty \\ \text{g.c.d.}(n,mN)=1}}^{\infty} \frac{1}{|mNz+n|^{2s}} \\
&= 2\zeta_N(2s)F_N(z,s).
\end{aligned}$$

On the other hand by applying the classical identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases},$$

we have

$$\begin{aligned}
S &= \sum_{m,n}{}' \left\{ \frac{1}{|mNz+n|^{2s}} \sum_{d|\text{g.c.d.}(n,N)} \mu(d) \right\} \\
&= \sum_{d|N} \left\{ \mu(d) \sum_{m,n}{}' \frac{1}{|mNz+n_1 d|^{2s}} \right\}
\end{aligned}$$

where $n_1 = \frac{n}{d}$. So

$$\begin{aligned} S &= \sum_{d|N} \left\{ \frac{\mu(d)}{d^{2s}} \sum_{m,n_1}{}' \frac{1}{|m\frac{N}{d}z + n_1|^{2s}} \right\} \\ &= \sum_{d|N} \left\{ \frac{\mu(d)}{d^{2s}} E\left(\frac{N}{d}z, s\right) \right\}. \end{aligned}$$

This completes the proof. □

We are ready to prove the main result of this chapter.

Theorem 6 (Rankin) *The Rankin-Selberg convolution $L(f\times g, s)$ has the following properties:*
(i) The series

$$L(f \times g, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

is absolutely and uniformly convergent for $Re(s) > 1$.
(ii) $L(f \times g, s)$ has a meromorphic continuation to the whole complex plane.
(iii) $L(f \times g, s)$ is analytic at $s = 1$ if $\langle f, g\rangle = 0$. Otherwise, it has a simple pole at point $s = 1$ with the residue

$$\begin{aligned} r &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})} \iint_{D_0(N)} \delta(f,g)dxdy \\ &= \frac{12(4\pi)^{k-1}}{N(k-1)!\prod_{p|N}(1+\frac{1}{p})} \langle f, g\rangle. \end{aligned}$$

(iv) Let

$$L(f \otimes g, s) = \zeta_N(2s)L(f \times g, s) = \zeta_N(2s)\sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the modified Rankin-Selberg convolution and for $Re(s) > 1$, let

$$\begin{aligned} \Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f\otimes g, s) \\ &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f\times g, s). \end{aligned}$$

Then both $L(f \otimes g, s)$ and $\Phi(s)$ are entire functions if $\langle f, g\rangle = 0$. Otherwise, if $N = 1$ they are analytic everywhere except that $L(f \otimes g, s)$ has a simple pole at point $s = 1$ and

$\Phi(s)$ has simple poles at points $s=0$ and 1, and if $N>1$ they are analytic everywhere except that $L(f\otimes g,s)$ has a simple pole at point $s=1$ and $\Phi(s)$ has a simple pole at points $s=1$.

(v) If $N=1$, then the function $\Phi(s)$ is invariant under the replacing of s by $1-s$, i.e.,

$$\Phi(s)=\Phi(1-s).$$

Proof *(i)* Suppose that $\sigma = Re(s)\geq 1+\delta>1$. By Deligne's bound, we know that $|a_f(n)|,|a_g(n)|\ll n^{\delta/4}$. So,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{a_f(n)\overline{a_g(n)}}{n^s}\right| &\ll \sum_{n=1}^{\infty}\frac{n^{\delta/2}}{n^{\sigma}}\\ &\leq \sum_{n=1}^{\infty}\frac{1}{n^{1+\delta/2}}\\ &< +\infty.\end{aligned}$$

This completes the proof of *(i)*.

(ii) & *(iv)* By Lemma 3 and Lemma 4, we have

$$\begin{aligned}\Phi(s) &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s}\Gamma(s)\Gamma(s+k-1)L(f\otimes g,s)\\ &= \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s}\Gamma(s)\zeta_N(2s)(4\pi)^{s+k-1}\iint_S y^s\delta(f,g)dxdy\\ &= (4\pi)^{k-1}\left(\frac{N}{\pi}\right)^{s}\Gamma(s)\zeta_N(2s)\iint_{D_0(N)} y^s\delta(f,g)F_N(z,s)dxdy.\end{aligned}$$

Applying Lemma 5 in the previous integral yields

$$\begin{aligned}\Phi(s) &= \frac{(4\pi)^{k-1}}{2}\left(\frac{N}{\pi}\right)^{s}\Gamma(s)\iint_{D_0(N)} y^s\delta(f,g)\sum_{d|N}\left(\frac{\mu(d)}{d^{2s}}E\left(\frac{N}{d}z,s\right)\right)dxdy\\ &= \frac{(4\pi)^{k-1}}{2}\iint_{D_0(N)}\delta(f,g)\sum_{d|N}\left(\frac{\mu(d)}{d^s}\left(\frac{Ny}{d\pi}\right)^{s}\Gamma(s)E\left(\frac{N}{d}z,s\right)\right)dxdy.\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\Phi(s) &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(N)} \delta(f,g) \sum_{d|N} \left(\frac{\mu(d)}{d^s} \xi \left(\frac{N}{d} z, s \right) \right) dxdy \\
&= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \left(\delta(f,g) \int_1^\infty \Theta(\omega)(\omega^{s-1} + \omega^{-s})\, d\omega \right) dxdy \\
&+ \frac{(4\pi)^{k-1}}{2s(s-1)} \sum_{d|N} \frac{\mu(d)}{d^s} \iint_{D_0(N)} \delta(f,g) dxdy. \qquad (1)
\end{aligned}$$

Note that the integral in the first summand of the right-hand side of (1) is dominated by a finite sum of integrals of the form

$$\iint_{D_0(N)} y^\lambda \delta(f,g) \left(\int_1^\infty e^{-a\omega} \omega^b d\omega \right) dxdy$$

for $\lambda \in \mathbb{R}$. These integrals are all convergent, because f and g vanish at all the cusps of $D_0(N)$. Therefore the first summand in 1 is an entire function of s. This proves *(ii)* and *(iv)*.

(iii) If we multiply both sides of (1) by $s-1$ and then let $s \to 1^+$, we get

$$\begin{aligned}
&\lim_{s\to 1^+} (s-1) \left(\frac{2\pi}{\sqrt{N}} \right)^{-2s} \Gamma(s)\Gamma(s+k-1)\zeta_N(2s) L(f \times g, s) \\
&= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d} \iint_{D_0(N)} \delta(f,g) dxdy
\end{aligned}$$

and therefore

$$\begin{aligned}
r &= \operatorname{Res}(L(f \times g, s), 1) \\
&= \frac{12(4\pi)^{k-1}}{N(k-1)! \prod_{p|N} (1 + \frac{1}{p})} \iint_{D_0(N)} \delta(f,g) dxdy.
\end{aligned}$$

This completes the proof of part *(iii)*.

(v) Let $N = 1$. We can simplify (1) to

$$\begin{aligned}
\Phi(s) &= \frac{(4\pi)^{k-1}}{2} \iint_{D_0(1)} \left(\delta(f,g) \int_1^\infty \Theta(\omega)(\omega^{s-1} + \omega^{-s}) \right) dxdyd\omega \\
&+ \frac{(4\pi)^{k-1}}{2s(s-1)} \iint_{D_0(1)} \delta(f,g) dxdy.
\end{aligned}$$

At a glance we realize that the right-hand side of this equality is invariant under the replacing of s with $1-s$. Therefore

$$\Phi(s) = \Phi(1-s).$$

In other words, $L(f \times g, s)$ satisfies the following functional equation

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s+k-1)\zeta_N(2s)L(f \times g, s)$$

$$= (2\pi)^{2s-2}\Gamma(1-s)\Gamma(k-s)\zeta_N(2-2s)L(f \times g, 1-s).$$

The proof of the theorem is complete. □

Exercise 7 *Without appealing to Deligne's bound show that $L(f \times g, s)$ is absolutely convergent for $Re(s) > 1$.*

Next we will study the Euler product of the Rankin-Selberg convolution of two modular L-functions. Let $f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n)e^{2\pi inz}$ be a cusp form for $\Gamma_0(N)$, and let $L_f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$ be its associated L-function. We know that $L_f(s)$ has an Euler product if and only if $f(z)$ is an eigenform. The next proposition will establish the Euler product of the modified Rankin-Selberg convolution of the modular L-functions associated to two eigenforms f and g. To derive the desired Euler product we need the following lemma.

Lemma 8 *Let f and g be two normalized eigenforms in $\Gamma_0(N)$, and let*

$$L_f(s) = \prod_{p|N} \left(1 - a_f(p)p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - \epsilon_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p p^{-s}\right)^{-1}$$

and

$$L_g(s) = \prod_{p|N} \left(1 - a_g(p)p^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - \delta_p p^{-s}\right)^{-1} \left(1 - \bar{\delta}_p p^{-s}\right)^{-1}$$

be their associated L-functions, where $\epsilon_p + \bar{\epsilon}_p = a_f(p)$, $\delta_p + \bar{\delta}_p = a_g(p)$ and $|\epsilon_p| = |\delta_p| = 1$. Then, for $Re(s) > 1$ and $p \nmid N$, we have the following identity

$$(1 - p^{-2s})^{-1} \sum_{k=0}^{\infty} \frac{a_f(p^k)a_g(p^k)}{p^{ks}}$$

$$= \left(1 - \epsilon_p\delta_p p^{-s}\right)^{-1} \left(1 - \epsilon_p\bar{\delta}_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p\delta_p p^{-s}\right)^{-1} \left(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)^{-1}.$$

Proof Let $p \nmid N$. We recall that the coefficients $a_f(n)$ and $a_g(n)$ satisfy the following:

$$a_f(p^k) = a_f(p)a_f(p^{k-1}) - a_f(p^{k-2}),$$

$$a_g(p^k) = a_g(p)a_g(p^{k-1}) - a_g(p^{k-2}).$$

Applying the above identities repeatedly yields

$$a_f(p^k)a_g(p^k) - a_f(p)a_f(p^{k-1})a_g(p)a_g(p^{k-1}) + \left(a_f(p)^2 + a_g(p)^2 - 2\right) a_f(p^{k-2})a_g(p^{k-2})$$

$$-a_f(p)a_f(p^{k-3})a_g(p)a_g(p^{k-3}) + a_f(p^{k-4})a_g(p^{k-4}) = 0. \quad (2)$$

Also by using the above relations between the coefficients $a_f(p)$, $a_g(p)$ and the complex units ϵ_p, δ_p, we have

$$\left(1 - \epsilon_p\delta_p p^{-s}\right)\left(1 - \epsilon_p\bar{\delta}_p p^{-s}\right)\left(1 - \bar{\epsilon}_p\delta_p p^{-s}\right)\left(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)$$

$$= 1 - a_f(p)a_g(p)p^{-s} + \left(a_f(p)^2 + a_g(p)^2 - 2\right) p^{-2s} - a_f(p)a_g(p)p^{-3s} + p^{-4s}. \quad (3)$$

Putting together (2) and (3), and following a tedious calculation, we arrive at

$$\left(1 - \epsilon_p\delta_p p^{-s}\right)\left(1 - \epsilon_p\bar{\delta}_p p^{-s}\right)\left(1 - \bar{\epsilon}_p\delta_p p^{-s}\right)\left(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)\sum_{k=0}^{\infty}\frac{a_f(p^k)a_g(p^k)}{p^{ks}}$$

$$= 1 - \frac{1}{p^{2s}},$$

which is equivalent to the statement of the lemma.
This completes the proof. □

Proposition 9 *The modified Rankin-Selberg convolution of the modular L-functions associated to two normalized eigenforms f and g has the following Euler product*

$$\begin{aligned} L(f \otimes g, s) &= \prod_{p \mid N}\left(1 - a_f(p)a_g(p)p^{-s}\right)^{-1} \\ &\times \prod_{p \nmid N}\left(1 - \epsilon_p\delta_p p^{-s}\right)^{-1}\left(1 - \epsilon_p\bar{\delta}_p p^{-s}\right)^{-1}\left(1 - \bar{\epsilon}_p\delta_p p^{-s}\right)^{-1}\left(1 - \bar{\epsilon}_p\bar{\delta}_p p^{-s}\right)^{-1}. \end{aligned}$$

Proof First of all we recall that the coefficients of eigenforms are multiplicative and real. So we have

$$L(f \otimes g, s) = \zeta_N(2s) \prod_{\text{all primes}} \left(\sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}} \right).$$

For $p \mid N$, since $a_f(p^k) = a_f(p)^k$ and $a_g(p^k) = a_g(p)^k$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{a_f(p^k) a_g(p^k)}{p^{ks}} &= \sum_{k=0}^{\infty} \frac{a_f(p)^k a_g(p)^k}{p^{ks}} \\ &= \left(1 - a_f(p) a_g(p) p^{-s}\right)^{-1}. \end{aligned}$$

Using this and applying the previous lemma, we attain the result. □