# Classical Analytic Theory of $L$-functions

## Lecture 2: Zeros of $L$-functions

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## 1 Zeros of $\Lambda(f,s)$

**Complex Analysis 1** *If the relation* $f(s)=O(e^{|s|})$ *does not hold for an entire function of order* 1*, then*

$$\sum_{\rho\neq 0}\frac{1}{|\rho|} \quad \text{is divergent,}$$

*and*

$$\sum_{\rho\neq 0}\frac{1}{|\rho|^{1+\epsilon}} \quad \text{is convergent,}$$

*where* $\rho$ *denotes zeros of the function and* $\epsilon>0$*. So in this case* $f(s)$ *has infinitely many zeros.*

Recall that $\Lambda(f,s)$ is a complete $L$-function in $\mathcal{IK}$ class.

**Exercise 2** *(i)* $\Lambda(f,s)$ *has infinitely many zeros.*
*(ii)* $\Lambda(f,s)$ *and* $L(f,s)\neq 0$ *on the half plane* $\Re(s)>1$*.*
*(iii) All zeros* $\rho$ *of* $\Lambda(f,s)$ *are in the critical strip* $0\leq\sigma\leq 1$*.*
*(iv) If* $\Re(\kappa_j)\geq 0$ *for* $1\leq j\leq d$*, then on the strip* $0<\Re(s)\leq 1$*, zeros of* $\Lambda(f,s)$ *and zeros of* $L(f,s)$ *coincide.*
*(v)* $\displaystyle\sum_{\rho\neq 0}\frac{1}{|\rho|^{1+\epsilon}}<\infty$*, where* $\rho$ *runs over zeros of* $\Lambda(f,s)$*.*
*(vi) If* $\rho$ *is a zero of* $\Lambda(f,s)$ *then* $1-\bar{\rho}$ *is also a zero of* $\Lambda(f,s)$*.*

## 2 Non-vanishing of $L$-functions on the line $\Re(s) = 1$

We define

$$\Lambda_f(n) = \begin{cases} \displaystyle\sum_{j=1}^{d} \alpha_j(p)^k \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}.$$

**Note** Please note that $\Lambda_f(n)$ is different from $\Lambda(f, s)$.

**Complex Analysis 3** ( **Logarithm of functions** ) *Let $f(s)$ be a function that is analytic and never $0$ on a simply connected region $A$. Then there is a function $g(s)$ analytic on $A$ and unique up to the addition of a constant multiple of $2\pi i$ such that $e^{g(s)} = f(s)$. Any $g(s)$ has formal properties similar to $\log f(s)$.*

**Exercise 4** *Let $\log z$ denote the principle branch of logarithm. Then $-\log(1-z)$ is analytic on $|z| < 1$ and it has the following Taylor's expansion*

$$-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$

*Use this fact to show that for $\sigma = \Re(s) > 1$, we have*

$$L(f, s) = \exp\left( \sum_p \sum_{k=1}^{\infty} \frac{\Lambda_f(p^k)}{k \log p \; p^{ks}} \right).$$

*Explain why it is reasonable to define for $\sigma > 1$*

$$\log L(f, s) = \sum_p \sum_{k=1}^{\infty} \frac{\Lambda_f(p^k)}{\log p^k \; p^{ks}}.$$

*Conclude that for $\sigma > 1$*

$$-\frac{L'}{L}(f, s) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}.$$

*Moreover show that for $\sigma > 2$*

$$\left| -\frac{L'}{L}(f, s) \right| \leq d\zeta'(\sigma - 1).$$

**Complex Analysis 5 (Landau's lemma)** *A Dirichlet series with non-negative coefficients has a singularity at its abscissa of convergence.*

**Lemma 6** *Let $f(s)$ be a complex function that satisfies the following:*
*(i) $f(s)$ is analytic on the half-plane $Re(s) > \sigma_0$;*
*(ii) $f(s)$ has a representation in the form*

$$f(s) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s}\right),$$

*with $c(n) \geq 0$ on the half-plane $\Re(s) > \sigma_1$ ($\sigma_1 > \sigma_0$).*
*Then $f(s) \neq 0$ for $Re(s) > \sigma_0$.*

**Proof** Let $\sigma_2$ be the abscissa of convergence of $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$. We claim that $\sigma_2 \leq \sigma_0$. To prove this let us assume that $\sigma_0 < \sigma_2 \leq \sigma_1$. Then for $\sigma > \sigma_2$ we have

$$f(\sigma) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma}\right). \tag{1}$$

Now since $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ is divergent at $\sigma_2$ and $f(s)$ is well defined at $\sigma_2$ the equality (1) shows that $f(\sigma_2) = 0$ and $\lim_{\sigma \to \sigma_2} \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} = -\infty$. (This is true since if $f(\sigma_2) \neq 0$, then $|f(s)| \neq 0$ on a neighborhood of $\sigma_2$ and so $\log f(s)$ gives a holomorphic continuation of $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ to the left of $\sigma_2$ which is a contradiction.) But $\lim_{\sigma \to \sigma_2} \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} = -\infty$ is impossible since $c(n) \geq 0$.

So $\sigma_2 \leq \sigma_0$. This shows that $\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ is convergent for $\Re(s) > \sigma_0$, and

$$f(s) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s}\right)$$

for $\Re(s) > \sigma_0$. So $f(s) \neq 0$ on $\Re(s) > \sigma_0$. □

**Theorem 7** (**Rankin (1939), Ogg (1969)** ) *Let $L(f,s)$ be an entire $L$-function. Let $L(f \otimes f, s)$ exists and it has a simple pole at $s = 1$. Then $L(f, 1+it) \neq 0$ for all real $t$.*

**Proof** Suppose that $L(f, 1+it_0) = 0$, and let

$$g(s) = \zeta(s)\ L(f, s+it_0)\ L(\bar{f}, s-it_0)\ L(f \otimes f, s).$$

It is clear that $g(s)$ is entire. Now note that for $Re(s) > 1$,

$$g_{\mathrm{unr}}(s) = \exp\left(\sum_{(p,q(f))=1} \sum_{k=1}^{\infty} \frac{|1+\sum_j \alpha_j^k p^{-kit_0}|^2}{kp^{ks}}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^s}\right)$$

where $c(n) \geq 0$. So, $g_{\mathrm{unr}}(s)$ satisfies the conditions of Lemma 6 with $\sigma_1 = 1$, and therefore $g_{\mathrm{unr}}(s)$ and $g(s) \neq 0$ everywhere. This is a contradiction since $g(-2) = 0$. □

## 3 More on zeros of $\Lambda(f, s)$

**Complex Analysis 8** (**Weierstrass, Hadamard** ) *Let $f$ be an entire function of order 1. Then*

$$f(s) = s^r e^{a+bs} \prod_{\rho \neq 0} (1 - \frac{s}{\rho}) e^{s/\rho},$$

*uniformly and absolutely on all compact subsets of $\mathbb{C}$, where $r$ is the order of the zero of $f$ at $s = 0$ and $\rho$ runs over zeros of $f$ different from $0$.*

As part of Weierstrass's theory we assume the legitimacy of any formal transformation of the above product formula.

**Theorem 9** *There exists constants $a = a(f)$ and $b = b(f)$ such that*

$$(s(1-s))^r \Lambda(f, s) = e^{a+bs} \prod_{\rho \neq 0,1} (1 - \frac{s}{\rho}) e^{s/\rho},$$

*where $\rho$ ranges over all zeros of $\Lambda(f, s)$ different from $0$ and $1$. This expansion is uniformly and absolutely convergent on compact subsets of complex plane. Moreover, the following identity is valid on any subset of complex plane that avoids zeros of $L(f, s)$.*

$$-\frac{L'}{L}(f, s) = \frac{1}{2} \log q(f) + \frac{\gamma'}{\gamma}(f, s) - b(f) + \frac{r}{s} + \frac{r}{s-1} - \sum_{\rho \neq 0,1} (\frac{1}{s-\rho} + \frac{1}{\rho}). \tag{2}$$

**Proof** These are consequences of Complex Analysis 8 and logarithmic differentiation. □

**Exercise 10** *Utilize the functional equation to show that*

$$\Re(b(f)) = -\sum_{\rho} \Re(\frac{1}{\rho}).$$

**Analytic Conductor (Iwaniec-Sarnak)** The *conductor of* $L(f,s)$ *at* $\infty$ is defined as

$$q_\infty(f,s) = \prod_{j=1}^{d}(|s+\kappa_j|+3).$$

The analytic conductor of $L(f,s)$ is defined as

$$Q(f,s) = q(f)q_\infty(f,s) = q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3).$$

We let

$$Q(f) = Q(f,0) = q(f)\prod_{j=1}^{d}(|\kappa_j|+3),$$

and similarly $q_\infty(f) = q_\infty(f,0)$.

**Lemma 11** $d \le \log q_\infty(f) \le \log Q(f)$.

**Proof** We have

$$q_\infty(f) = \prod_{j=1}^{d}(|\kappa_j|+3) \ge 3^d.$$

The result follows by taking the logarithm from both sides of this inequality. □

**Complex Analysis 12**

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right),$$

*is valid as* $|s| \to \infty$, *in the angle* $-\pi+\delta < \arg s < \pi-\delta$, *for any fixed* $\delta > 0$.

**Estimation of the gamma factors** By employing Complex Analysis 9 we can estimate the gamma factors as follows. For $\Re(s) > 1$ we have

$$\frac{\gamma'}{\gamma}(f,s) \ll d + \log q_\infty(f,s) + \sum_{|s+\kappa_j|<1}\frac{1}{|s+\kappa_j|} \tag{3}$$

**Lemma 13 (Main Identity)** *On the half-plane* $\Re(s) > 1$

$$\sum_{\rho\neq 0,1}\Re(\frac{1}{s-\rho}) = \Re(\frac{r}{s-1}) + \Re(\frac{r}{s}) + O(\sum_{|s+\kappa_j|<1}\frac{1}{|s+\kappa_j|}) + \Re(\frac{L'}{L}(f,s)) + O(\log Q(f,s)),$$

*where* $\rho$ *ranges over all zeros of* $\Lambda(f,s)$ *different from* 0 *and* 1.

**Proposition 14** *Let $\rho = \beta + i\gamma$ denote the zeros of $\Lambda(f,s)$. Then*

$$\sum_{\rho} \frac{1}{1+(T-\gamma)^2} \ll \sum_{\rho} \Re(\frac{1}{3+iT-\rho}) \ll \log Q(f, iT).$$

*The implied constant depends only on $r$.*

**Note** You should not confuse the ordinate $\gamma$ with the factors $\gamma(f,s)$.

**Proof** Let $s = 3 + iT$ in the Main Identity. The result follows. ☐

**Corollary 15** *Let $N(f,T)$ be the number of zeros of $\rho = \beta + i\gamma$ of $L(f,s)$ such that $0 \le \beta \le 1$ and $0 \le \gamma \le T$. Then*

$$N(f, T+1) - N(f,T) = O(\log Q(f, iT)).$$

*The implied constant depends only on $r$.*

**Proof** Since zeros of $L(f,s)$ and $\Lambda(f,s)$ on the critical strip are basically the same (at most finitely many exceptions). From the previous proposition, we have

$$N(f,T+1) - N(f,T) = \sum_{\substack{\rho \\ T<\gamma\le T+1}} 1 \le \sum_{\rho} \frac{2}{1+(T-\gamma)^2} \ll \log Q(f, iT).$$

☐

**Note** One can show that

$$N(f,T) = \frac{d}{2\pi} T \log T + cT + O(\log T),$$

as $T \to \infty$. Here $c$ is a constant depends on $L(f,s)$.

# 4 A zero-free region

**Lemma 16** $Q(f,s) \le Q(f)(|s|+3)^d$.

**Proof** We have

$$Q(f,s) = q(f) \prod_{j=1}^{d} (|s+\kappa_j| + 3) \le q(f) \prod_{j=1}^{d} (|k_j|+3)(|s|+3) = Q(f)(|s|+3)^d.$$

☐

**Exercise 17** *Assume that for any ramified prime (i.e. $p$ such that $(p, q(f)) \neq 1$) we have $|\alpha_j(p)| \leq p/2$. Then show that for $\sigma > 1$*

$$\Re(\frac{L'_{\mathrm{r}}}{L_{\mathrm{r}}}(f, \sigma)) = O(d \log q(f)).$$

**Lemma 18 (Goldfeld, Hoffstein and Lieman (1994) via de la Vallée Poussin (1899))** *Suppose that $\Re(\Lambda_f(n)) \geq 0$ for $(n, q(f)) = 1$. Suppose that $\Re(\kappa_j) > -1/2$ and at ramified primes $|\alpha_j(p)| \leq p/2$. Let $r$ be the order of $L(f, s)$ at $s = 1$. Then*
*(i) $L(f, 1) \neq 0$. In other words $r$ is non-negative.*
*(ii) There exists an effective constant $c > 0$, depending only on $r$, such that $L(f, s)$ has at most $r$ real zeros in the interval*

$$\sigma \geq 1 - \frac{c}{d(r+1) \log Q(f)}.$$

**Proof** Let $1 < \sigma < \frac{3}{2}$. Then from the previous lemma and Lemma 16

$$\sum_{\rho \neq 0,1} \Re(\frac{1}{\sigma - \rho}) = \frac{r}{\sigma - 1} + \frac{r}{\sigma} + O(\sum_{|\sigma + \kappa_j| < 1} \frac{1}{|\sigma + \kappa_j|}) + \Re(\frac{L'}{L}(f, \sigma)) + O(\log Q(f)),$$

where $\rho$ ranges over all zeros of $\Lambda(f, s)$ different from 0 and 1. Now note that

$$\Re(\frac{1}{\sigma - \rho}) > 0, \quad \Re(\frac{L'_{ur}}{L_{ur}}(f, \sigma)) \leq 0, \quad \text{and} \quad \Re(\frac{L'_{\mathrm{r}}}{L_{\mathrm{r}}}(f, \sigma)) = O(d \log q(f)).$$

So if $\beta_j$'s are the zeros of $L(f, s)$ in the interval $[\frac{1}{2}, 1)$, there exists a constant $c_1$ (depending only on $r$) such that

$$\sum_j \frac{1}{\sigma - \beta_j} \leq \frac{r}{\sigma - 1} + c_1 d \log Q(f).$$

(Note that since $\Re(\kappa_j) > -1/2$, $\beta_j$'s are also zeros of $\Lambda(f, s)$). Now if $\sigma \to 1^+$ this inequality shows that $r$ cannot be negative. so $r \geq 0$.
Now for $\delta, c > 0$ let $\sigma = 1 + \delta/d \log Q(f)$ and $m$ be the number of zeros of $L(f, s)$ in the interval $(1 - c/d(r+1) \log Q(f), 1)$. Then from the previous inequality, we have

$$m \leq (\delta + \frac{c}{r+1})(\frac{r}{\delta} + c_1) = r + \delta c_1 + \frac{c}{\delta}(\frac{r}{r+1}) + \frac{cc_1}{r+1}.$$

Now let $\delta < c_1^{-1}$, then we can choose $c$ small enough such that $m \leq r$. The proof now is complete. □

Let $L(f,s)$ be an entire $L$-function of degree $d$ with at least one non-real coefficient (i.e $L(f,s)$ is not self-dual). Suppose that the Ranking-Selberg $L$-functions $L(f\otimes f,s)$ and $L(f\otimes\bar{f},s)$ exist. Also suppose that $\Re(\mu_{i,j})>-\frac{1}{2}$. (Note that this implies that $\Re(\kappa_j)>-1/4$.) Moreover we assume that the pole of $L(f\otimes f,s)$ at $s=1$ is simple and $L(f\otimes\bar{f},s)$ is entire. Finally we assume that the local parameters $\alpha_j(p)$ of $L(f,s)$, $L(f\otimes f,s)$ and $L(f\otimes\bar{f},s)$ at the ramified primes satisfy in the inequality $|\alpha_j(p)|\leq p/2$.

**Exercise 19** $Q(f\otimes f)\ll Q(f)^{2d}$.

The next theorem establishes a zero free region for such $L$-functions.

**Theorem 20** *There exists an absolute constant* $c>0$ *such that* $L(f,s)$ *has no zeros in the region*

$$\sigma\geq 1-\frac{c}{d^4\log\left(Q(f)(|t|+3)\right)}.$$

**Proof** For $t\in\mathbb{R}$ , we let

$$L(g,s)=\zeta(s)L(f,s+it)^2L(\bar{f},s-it)^2L(f\otimes\bar{f},s+2it)L(\bar{f}\otimes f,s-2it)L(f\otimes f,s)^2.$$

It is clear that $L(g,s)$ is an $L$-function of degree $(1+2d)^2$. By employing Exercise 19 we have

$$Q(g)\ll Q(f)^{4+8d}(|t|+3)^{6d^2}.$$

Also we have $\Lambda_g(n)\geq 0$ for any $n$ coprime to $q(g)$ (or $q(f)$).
Now let $\rho=\beta+i\gamma$ be a zero of $L(f,s)$ with $\beta\geq 1/2$. In $L(g,s)$ let $t=\gamma$. Then $L(g,s)$ has a pole of order at most 3 at $s=1$ and a zero of at least 4 at $s=\beta$. By Lemma 18 it is clear that

$$\beta<1-\frac{c}{d^2\log Q(g)}<1-\frac{c'}{d^4\log\left(Q(f)(|t|+3)\right)},$$

for some absolute constants $c>0$ and $c'>0$. The proof now is complete. ☐

Let $L(f,s)$ be an entire $L$-function of degree $d$ with real coefficients (i.e $L(f,s)$ is self-dual). Suppose that the Ranking-Selberg $L$-functions $L(f\otimes f,s)$ exists. Also we suppose that $\Re(\mu_{i,j})>-\frac{1}{2}$. (Note that this implies that $\Re(\kappa_j)>-1/4$.) Moreover we assume that the pole of $L(f\otimes f,s)$ at $s=1$ is simple. Finally we assume that the local parameters $\alpha_j(p)$ of $L(f,s)$ and $L(f\otimes f,s)$ at the ramified primes satisfy in the inequality $|\alpha_j(p)|\leq p/2$. The next theorem establishes an almost zero free region for such $L$-functions.

**Theorem 21** *There exists an absolute constant* $c > 0$ *such that* $L(f, s)$ *has no zeros in the region*

$$\sigma \geq 1 - \frac{c}{d^4 \log\left(Q(f)(|t| + 3)\right)},$$

*except possibly for one simple real zero* $\beta_f < 1$*.*

**Proof** The proof is the same as the previous theorem. The only difference is that if $t = \gamma = 0$, then $L(g, s)$ has a pole of order at most 5 (in fact exactly 5) at $s = 1$ and a zero of at least 4 at $s = \beta$. So by Lemma 18 there is an absolute constant $c > 0$ such that $L(f, s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c}{d^4 \log\left(Q(f)(|t| + 3)\right)},$$

except possibly for one simple real zero $\beta_f$. Since $L(f, s)$ is non-vanishing on the line $\Re(s) = 1$, we have $\beta_f < 1$. ☐

**Note** The possible simple real zero $\beta_f$ of $L(f, s)$ is called the *exceptional zero* or the *Siegel zero*.