Two Remarks on Ancient Greek Geometry

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These two remarks are really about the way modern mathematicians interpret the ancient theories. Furthermore, they are really about the interpretation of the theories that follow the publication of Euclid's *Elements*, since these are the theories about which we have the most knowledge and which have had the most influence on our own mathematics.

1 Why Euclidean Geometry?

Modern physical theories tell us that space is not euclidean. Yet, to most people, euclidean geometry seems most natural. Furthermore, most mathematical theories of non-euclidean geometry refer back to euclidean geometry in one way or another. For example, in differential geometry, the differential areas are almost always approximately Euclidean. Why is this?

I want to suggest here a reason rooted in our biological programming.

I base this on two technical results. One, mentioned by Herbert Busemann [2], is an idea of the physicist Helmholtz about the mobility of rigid objects. As Busemann puts it (on p. 51, his emphasis),

> ... Helmholtz emphasized that *the mobility of rigid objects, the fact that we recognize an object independently of its position in space, is one of our basic physical experiences.* He therefore put, and partially solved, the mathematical problem of finding those spaces or geometries in which figures can be moved freely.
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> The answer to this problem is that euclidean and non-euclidean geometries have that property and that no other geometry does.

By "non-euclidean geometries", Busemann means hyperbolic and elliptical geometry.

The mobility of rigid objects is now recognized as one of the things evey normal human child learns in infancy, and this learning appears to be part of our biological programming. It is true that modern physical theories tell us that it is not strictly true; objects actually change shape as they move. But we are programmed not to think about these changes. This would seem to commit us to one of three geometries: euclidean, hyperbolic, and elliptical.

The second technical result is John Wallis' proof that, in the presence of the axioms common to euclidean and non-euclidean geometry, the existence of triangles that are similar but not congruent is equivalent to the parallel postulate. (See Lanczos [7, p.

59].) The significance of this is that we are all used to thinking in terms of exact scale models, and this is true of every human culture of which I have ever heard.[1] But scale models require figures that are the same shape but different sizes, and are thus possible only if there exist triangles which are similar but not congruent. Hence, if we start with one of euclidean, hyperbolic, or elliptical geometry, Wallis' result and our use of scale models together imply that the geometry is euclidean.

In my opinion, these results strongly suggest that the programming with which we are all born tend to direct us to euclidean geometry. Of course there have been many cultures that have never taken up a systemmatic study of geometry, but I think that cultures that do will be induced by this programming to study euclidean geometry first.

2 Magnitudes: countable or uncountable?

In modern mathematics, what the Greeks called magnitues are treated as real numbers. This is illustrated by the fact that in many of our elementary textbooks, students are told that the real numbers can be obtained from the rational numbers by representing the points on a line which cannot be represented by rational numbers. However, this is not the only way to do geometry: it is standard to do geometry with the surd field (obtained from the rational numbers by closing under square roots and the field operations) to prove that certain constructions cannot be accomplished with a ruler and compass alone (see [5]). The surd field is clearly not enough for all the constructions done by the ancient Greeks, but since it is a countable field, it raises the question of whether there is a countable subfield of the real numbers which is adequate for the ancient Greek approach to magnitudes.

Actually, the ancient Greek conceptions are more different from those of modern mathematicians than many people realize. Ivor Grattan-Guinness [4] discusses how different were the ancient Greek conceptions of number and magnitude in a critique of the tendency to regard theorems from Euclid's *Elements* as algebraic in nature. In Euclid, numbers and magnitudes are distinct, and there are different kinds of magnitudes. Here is a table of the different kinds taken from [4, p. 363].

Kind	Straight	Curved
lines	straight (line)	planar curved (arc of circle)
regions	planar rectilinear (rectangle)	planar curvilinear (segment of a circle)
surfaces	spatial rectilinear (pyramid)	spatial curvilinear (sphere)
solids	rectilinear (cube)	curvilinear (hemisphere)
angles	planar	solid planar

Furthermore, although magnitudes of the same kind can be added and subtracted (provided the smaller is subtracted from the larger), and although magnitudes can be multiplied by numbers, magnitudes are never multiplied by other magnitudes. (Ruler and compass constructions for multiplying and dividing linear lengths were first given

[1]This makes it reasonable to believe that this is also part of our biological programming.

by Descartes in [3].) Thus, for the ancient Greeks, magnitudes did not form what we call a field.[2]

However, it makes sense to talk about the fields we can use today to represent magnitudes. Furthermore, it makes sense to talk about using one field for all magnitues, because we are used to relating magnitudes that the Greeks regarded as being of different kinds. Thus, we can compare lines and regions by comparing the areas of figures to those of rectangles one of whose sides is a unit[3]; we can compare (rectilinear) surfaces to regions by taking each face and comparing it to a plane region and adding; we can compare solids to regions by comparing volumes to rectangular solids one of whose sides is a unit; and we can compare angles to lengths by comparing a given angle to the arc subtended in the unit circle. The techniques of calculus can be used to relate straight magnitudes to curved ones. Thus, for us, it is very natural to regard magnitudes as a field. It is now common to use the real numbers for this field.

But, it is not totally clear to me that the ancient Greek approach requires this. If we think, for example, of Cantor's proof that the real numbers are uncountable, we find that Cantor is associating real numbers with infinite decimals, which are completed infinities. From Aristotle on, most ancient Greek mathematicians seem to have rejected completed infinities (see [1, pp. 64f]).[4] A similar conclusion applies to Cauchy sequences.

On the other hand, Dedekind's axiom of continuity does not seem so obviously to involve completed infinities. This axiom is most easily stated in terms of the relation $A - B - C$ between points, which means that B is on the line segment joining A and C and is not one of the endpoints of the segment. Then the axiom is as follows:

If all the points of a straight line be divided into two classes $\mathcal{A}$ and $\mathcal{B}$ with the properties that all points of the line are in either $\mathcal{A}$ or $\mathcal{B}$ and, for any points A of $\mathcal{A}$ and B of $\mathcal{B}$, if C is any other point, then $A - B - C$ implies that C is in $\mathcal{B}$ whereas $B - A - C$ implies that C is in A, then there is a unique point D of the line which divides the points in $\mathcal{A}$ from the points in $\mathcal{B}$ (in the sense that for any points E of $\mathcal{A}$ and F of $\mathcal{B}$, $E - D - F$).

To us, this implies that there is a point on the line for each real number. And it would also seem that the ancient Greeks would have had to accept the statement of the axiom if it had been expressed in a form they could understand. But can we get from this definition to a need for all real numbers without completed infinities?

Consider the modern proof by Edwin E. Moise [8, pp. 483–484] that the set of real numbers is uncountable. This proof can easily be transferred to points on a straight line, and involves nested intervals. Take a reference point O (to play the role of 0) and a second reference point I to play the role of 1. An interval is just a line segment from the line specified by its endpoints (which are on the line). A *nested sequence of intervals* is a sequence $\overline{A_1B_1}, \overline{A_2B_2}, \ldots$ where each $\overline{A_{i+1}B_{i+1}}$ is contained in $\overline{A_iB_i}$. If

[2] The notions of magnitude found in Euclid seem to have developed after the discovery of the incommensurability of the side and diagonal of a square; see Knorr [6].

[3] The ancient Greeks only compared areas to squares.

[4] Archimedes was something of an exception. He accepted actual infinities in his book "On Method", but this was only to *discover* theorems, not to *prove* them. He also accepted a line segment equal in length to π.

we write $A < B$ to mean that $A-O-I$ and either $A-B-O$ or $B=O$ or $A-O-B$, or $A=0$ and either $A-B-I$ or $B=I$ or $A-I-B$, or neither $A-O-I$ nor $A=O$ and $O-A-B$, then we have the usual ordering relation, and the continuity axiom can be used to prove that the intersection of a nested sequence of intervals is not empty; this is done by defining a cut to be $\mathcal{A}, \mathcal{B}$ where $\mathcal{A}$ consists of all the A_i and any point "less than" any A_i and $\mathcal{B}$ to be all points not in $\mathcal{A}$. The next step is to suppose that the points of the line can be numbered in a sequence $X_1, X_2, \ldots, X_n, \ldots$. Then in modern terms a nested sequence of intervals $\overline{A_1 B_1}, \overline{A_2 B_2}, \ldots$ can be defined so that the interval $\overline{A_i B_i}$ does not contain the point X_i. It follows that the intersection of this nested sequence of intervals is not empty, but the point that is in the intersection cannot be any of the X_i, so the sequence $X_1, X_2, \ldots, X_n, \ldots$ cannot contain all points from the line. But how would the ancient Greeks have accepted this definition? To them it would seem to require infinitely many free choices. Given the controversy over the axiom of choice early in this century, it seems unlikely that the ancient Greeks would have accepted this.

However, the proof can be recast so as not to require such choices. In the second part of the proof, let $\overline{A_1, B_1}$ be $\overline{OI}$ if X_1 is not in $\overline{OI}$, and let it be $\overline{JO}$, where J is one unit away from O and $J-O-I$, otherwise. Now, assuming that $\overline{A_i B_i}$ is defined, let C_i be its midpoint; then $\overline{A_{i+1}, B_{i+1}}$ will be $\overline{A_i, C_i}$ unless X_{i+1} occurs in $\overline{A_i, C_i}$ and is $\overline{C_i, B_i}$ otherwise. Now, the nested sequence of intervals is defined by a fixed rule so that X_i does not occur in $\overline{A_i B_i}$, and I think this definition would have been accepted by the ancient Greeks. However, if we try to apply to this particular nested sequence of intervals the proof that its intersection is nonempty, we into a problem: the division of the line into two classes $\mathcal{A}$ and $\mathcal{B}$ is not complete until an infinite number of steps have been completed. Even Archimedes, who allowed completed infinities in his *Method* did not use infinite constructions in which each step depends on the result of a previous step. I strongly suspect that even Archimedes would have rejected a construction requiring the completion of an infinite number of steps in order.[5] For this reason, I do not believe any of the ancient Greek mathematicians after Aristotle would have accepted this proof. Note that what they would not have accepted is the proof that certain points exist, so for them a line may well have had fewer points than it does for us.

The question then arises: Can we find a countable field that can be used to analyze the ancient Greek magnitudes?

One possible approach to answering our question is to assume that the real numbers needed for ancient Greek mathematics would be among those for which we have names. This leads to the following suggestion for a field for magnitudes: start with the algebraic numbers, add π and e, and then close under the exponential, logarithmic, and trig functions and the field operations. This field is countable, and includes all real numbers with commonly used names.

There is a trivial sense in which this approach does lead to a countable field

[5]The unwillingness to accept completed infinities is somewhat like modern constructivism (for example, intuitionism), but there is a major difference: the ancient Greeks would have accepted the decidability of equality of points, but modern constructivists reject the decidability of equality of real numbers.

for magnitudes: if the field defined in the previous paragraph is not sufficient for all constructions carried out by the ancient Greeks, there are only a finite number of constructions for which it fails, and each of these constructions is finite. Therefore, by adding a finite number of new elements or closing under a finite number of additional functions will lead to a field adequate for all constructions that the ancient actually carried out.

But this is not a real answer to the original question, for we want a field adequate for all constructions the ancient Greeks actually did and, in addition, all constructions to which their methods would have led them without accepting completed infinities. And whether the field defined above is adequate for this purpose, I do not know.

What I do know is that ancient Greek mathematics is more different from modern mathematics than many people realize.

References

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