On Normalizing Disjunctive Intermediate Logics*

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Abstract

In this paper it is shown that every intermediate logic obtained from intuitionistic logic by adding a disjunction can be normalized. However, the normalization procedure is not as complete as that for intuitionistic and minimal logic because some results which usually follow from normalization fail, including the separation property and the subformula property.

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By a *disjuctive intermediate logic* I mean a system of logic obtained by adding to intuitionistic logic an axiom scheme of the form

$$C_1(A_1, A_2, \ldots, A_m) \vee C_2(A_1, A_2, \ldots, A_m) \vee \ldots \vee C_n(A_1, A_2, \ldots, A_m),$$

where $C_i(A_1, A_2, \ldots, A_m)$ is a formula scheme in which $A_1, A_2, \ldots, A_m$ occur as parameters.

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Some examples of disjunctive intermediate logics are given by Umezawa [9] as follows:

(M) $\neg\neg A \vee \neg A$.

(P_n) $(A_1 \supset A_2) \vee (A_1 \supset A_3) \vee \ldots \vee (A_1 \supset A_n) \vee (A_2 \supset A_1) \vee (A_2 \supset A_3) \vee \ldots \vee (A_2 \supset A_n) \vee \ldots \vee (A_n \supset A_1) \vee (A_n \supset A_2) \vee \ldots \vee (A_n \supset A_{n-1})$, where $n \geq 2$ and, for any $(A_i \supset A_j)$, $i \neq j$. A special case is (P_2), when the axiom is $(A_1 \supset A_2) \vee (A_2 \supset A_1)$, and the logic is also known as (LC).

(R_n) $A_1 \vee (A_1 \supset A_2) \vee (A_2 \supset A_3) \vee \ldots \vee (A_{n-1} \supset A_n) \vee \neg A_n$, where $n \geq 2$.

(ME) $(\forall x)\neg\neg A(x) \vee (\exists x)\neg A(x)$.

(MEK°) $\neg\neg(\forall x)A(x) \vee (\exists x)\neg A(x)$.

(DP_2) $(\forall x)(A \supset B(x)) \vee (\exists x)(B(x) \supset A)$.

(FP_2) $(\exists x)(\forall y)(A(x) \supset B(y)) \vee (\exists y)(\forall x)(B(y) \supset A(x))$.

(GP_2) $(\exists y)(\forall x)(A(x) \supset B(y)) \vee (\exists x)(\forall y)(B(y) \supset A(x))$.

(FGP_2) $(\exists x)(\exists u)(\forall y)(\forall v)(A(x,v) \supset B(y,u)) \vee (\exists y)(\exists v)(\forall x)(\forall u)(B(y,u) \supset A(x,v))$.

(ER_n) $(\forall x)A_1(x) \vee (\exists x)(\forall y)(A_1(x) \supset A_2(y)) \vee \ldots \vee (\exists x)(\forall y)(A_{n-1}(x) \supset A_n(y))) \vee (\exists x)\neg A_n(x)$.

In addition, classical logic can be formulated in this form as follows:

(K) $\neg A \vee A$.

López-Escobar [7] gives another example:

(LIC_n) $(A_1 \supset A_2) \vee (A_2 \supset A_3) \supset \ldots \supset (A_{n-1} \supset A_n) \vee (A_n \supset A_1)$, where $n \geq 2$. Note that (LC) is also (LIC_2),

and he also points out that classical logic can be axiomatized by adding to intuitionistic logic the axiom

(C) $(A \supset B) \vee A$.

(López-Escobar is considering only the implication fragments of these logics, but his definitions apply to full predicate logics.)

The purpose of this paper is to look at natural deduction versions of these systems and to prove a normalization result for them. The normalization result obtained does not imply all of the results of normalization for classical or intuitionistic logic. In particular, the subformula property fails. Arnon Avron [1] has used the method of *hypersequents* to give a formulation that does satisfy the subformula property for one of the logics considered here, the logic (LC).

Since the intuitionistic rule

$$\frac{\perp}{A}\ \perp_{\mathrm{I}}$$

is not used in any essential way in obtaining these rusults, they also hold if the axioms in question are added to minimal logic instead of intuitionistic logic. If (K) is added to LM, the result is the system which Curry [3] called (in a sequent L-version) LD. If (C) is added instead, the system obtained is the system called LE in [4], which is due to Paul Bernays [2] and first extensively studied by Saul Kripke [6].[1]

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1 The Natural Deduction Formulation

First, let us recall the system TM[2] of minimal predicate logic. It has no axioms, and its rules are as follows:

$$\wedge\mathrm{I}\ \frac{A_1 \qquad A_2}{A_1 \wedge A_2} \qquad\qquad \wedge\mathrm{E}\ \frac{A_1 \wedge A_2}{A_i}$$

$$\vee\mathrm{I}\ \frac{A_i}{A_1 \vee A_2} \qquad\qquad \vee\mathrm{E}\ \frac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

[1]See [4, p. 260, p. 306]. The system was also studied by Kanger [5] before Kripke's extensive study.

[2]This name is due to Curry; see [4, p. 280]. Actually, Curry reserved the name TM for propositional logic only, using the name TM* for the predicate logic, but since I am not considering the popositional case separately, I will drop the superscript asterisk.

$\supset$I $\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$ $\supset$E $\dfrac{A \supset B \quad A}{B}$

$\forall$I $\dfrac{A(c)}{(\forall x)A(x)}$ $\forall$E $\dfrac{(\forall x)A(x)}{A(t)}$

$\exists$I $\dfrac{A(t)}{(\exists x)A(x)}$ $\exists$E $\dfrac{(\exists x)A(x) \quad \begin{matrix}[A(c)]\\ C\end{matrix}}{C}$

where in rules $\wedge$E and $\vee$I, $i = 1$ or $i = 2$; in rules $\forall$E and $\exists$I, t is any term; and in rules $\forall$I and $\exists$E, c is an *eigenvariable*, which is a variable which does not occur free in any undischarged assumption (or, in the case of $\exists$E, in C). Also, $\neg A$ is defined to be $A \supset \bot$, so that negation satisfies the derived rules

$\neg$I $\dfrac{\begin{matrix}[A]\\ \bot\end{matrix}}{\neg A}$ $\neg$E $\dfrac{\neg A \quad A}{\bot}$

Recall also that the system TJ of intuitionistic logic is obtained from TM by adding the rule

$\bot_{\mathrm{I}}$ $\dfrac{A}{\bot}$

Now, let us write $C_i(\vec{A})$ for $C_i(A_1, A_2, \ldots, A_m)$. Then the axiom of a disjunctive intermediate logic has the form

$$C_1(\vec{A}) \vee C_2(\vec{A}) \vee \ldots \vee C_n(\vec{A}). \tag{1}$$

For the systems he considers, López-Escobar [7] proposes replacing axiom (1) by the rule

$$\dfrac{\begin{matrix}[C_1(\vec{A})]\\ C\end{matrix} \quad \begin{matrix}[C_2(\vec{A})]\\ C\end{matrix} \quad \ldots \quad \begin{matrix}[C_n(\vec{A})]\\ C\end{matrix}}{C}\ \mathrm{Ds}$$

Theorem 1 *Adding axiom (1) to TM (respectively TJ) is equivalent to adding rule* Ds *to TM (respectively TJ).*

Proof Suppose that we have added rule Ds to TM (or, for that matter TJ), and suppose we are given deductions

$$\begin{matrix} C_1(\vec{A}) \\ \mathcal{D}_1 \\ C \end{matrix}\;, \quad \begin{matrix} C_2(\vec{A}) \\ \mathcal{D}_2 \\ C \end{matrix}\;, \quad \ldots, \quad \begin{matrix} C_n(\vec{A}) \\ \mathcal{D}_n \\ C \end{matrix}\;.$$

Then, writing C_i for $C_i(\vec{A})$, E_2 for $C_2 \vee E_3$, E_3 for $C_3 \vee E_4$, $\ldots$, E_{n-1} for $C_{n-1} \vee C_n$, we can proceed as follows:

$$\dfrac{C_1 \vee E_2 \qquad \begin{matrix} \overset{1}{[C_1]} \\ \mathcal{D}_1 \\ C \end{matrix} \qquad \dfrac{\overset{2}{[C_2 \vee E_3]} \qquad \begin{matrix} \overset{3}{[C_2]} \\ \mathcal{D}_2 \\ C \end{matrix}\ \ldots \qquad \begin{matrix} \dfrac{\overset{2n-2}{[C_{n-1} \vee C_n]} \qquad \begin{matrix} \overset{2n-1}{[C_{n-1}]} \\ \mathcal{D}_{n-1} \\ C \end{matrix} \quad \begin{matrix} \overset{2n}{[C_n]} \\ \mathcal{D}_n \\ C \end{matrix}}{C}\ \vee\mathrm{E} - (2n-1), 2n \\ \vdots \\ C \end{matrix}}{C}\ \vee\mathrm{E} - 3, 4}{C}\ \vee\mathrm{E} - 1, 2$$

Conversely, suppose we have added rule Ds to TM. Then we can deduce axiom (1) as follows:

$$\dfrac{\dfrac{\overset{1}{[C_1]}}{C_1 \vee C_2 \vee \ldots \vee C_n}\ \text{repeated} \vee \mathrm{E} \qquad \ldots \qquad \dfrac{\overset{n}{[C_n]}}{C_1 \vee C_2 \vee \ldots \vee C_n}\ \text{repeated} \vee \mathrm{E}}{C_1 \vee C_2 \vee \ldots \vee C_n}\ \mathrm{Ds} - 1, 2, \ldots n$$

■

For the rest of the paper, we will assume that our disjunctive intermediate logic is obtained from TJ by adding rule Ds.

Definition 1 The *system TI* will be obtained from TJ by adding the rule Ds.

2 The Normalization Result

Recall that the deduction reduction steps for TJ (and also TM) are as follows:

$\wedge$-reduction steps:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ A_1 & A_2\end{matrix}}{A_1 \wedge A_2}\,\wedge\mathrm{I}}{\begin{matrix}A_i\\ \mathcal{D}_3\end{matrix}}\,\wedge\mathrm{E} \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_i\\ A_i\\ \mathcal{D}_3\end{matrix}$$

$\vee$-reduction steps:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ A_i\end{matrix}}{A_1 \vee A_2}\,\vee\mathrm{I} \quad \begin{matrix}1\\ [A_1]\\ \mathcal{D}_1\\ C\end{matrix} \quad \begin{matrix}2\\ [A_2]\\ \mathcal{D}_2\\ C\end{matrix}}{\begin{matrix}C\\ \mathcal{D}_3\end{matrix}}\,\vee\mathrm{E}-1,2 \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_0\\ A_i\\ \mathcal{D}_i\\ C\\ \mathcal{D}_3\end{matrix}$$

$\supset$-reduction steps:

$$\dfrac{\dfrac{\begin{matrix}1\\ [A]\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\,\supset\mathrm{I}-1 \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{\begin{matrix}B\\ \mathcal{D}_3\end{matrix}}\,\supset\mathrm{E} \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_2\\ A\\ \mathcal{D}_1\\ B\\ \mathcal{D}_3\end{matrix}$$

$\forall$-reduction steps:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1(c)\\ A(c)\end{matrix}}{(\forall x)A(x)}\,\forall\mathrm{I}}{\begin{matrix}A(t)\\ \mathcal{D}_3\end{matrix}}\,\forall\mathrm{E} \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_1(t)\\ A(t)\\ \mathcal{D}_3\end{matrix}$$

$\exists$-reduction steps:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ A(t)\end{matrix}}{(\exists x)A(x)}\,\exists\mathrm{I} \quad \begin{matrix}1\\ [A(c)]\\ \mathcal{D}_2(c)\\ C\end{matrix}}{\begin{matrix}C\\ \mathcal{D}_3\end{matrix}}\,\exists\mathrm{E}-1 \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}_1\\ A(t)\\ \mathcal{D}_2(t)\\ C\\ \mathcal{D}_3\end{matrix}$$

$\vee R$-reduction steps: If R is an E-rule with C as its major (left) premise and $(\mathcal{D}_3)$ as the deduction(s) of its minor premises, if any, then

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ A_1 \vee A_2\end{array}\quad \begin{array}{c}1\\ [A_1]\\ \mathcal{D}_1\\ C\end{array}\quad \begin{array}{c}2\\ [A_2]\\ \mathcal{D}_2\\ C\end{array}}{C}\,\vee\mathrm{E}-1,2 \qquad (\mathcal{D}_3)}{\begin{array}{c}E\\ \mathcal{D}_4\end{array}}\,R$$

reduces to

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ A_1 \vee A_2\end{array}\quad \dfrac{\begin{array}{c}1\\ [A_1]\\ \mathcal{D}_1\\ C\end{array}\quad (\mathcal{D}_3)}{E}\,R \quad \dfrac{\begin{array}{c}2\\ [A_2]\\ \mathcal{D}_2\\ C\end{array}\quad (\mathcal{D}_3)}{E}\,R}{\begin{array}{c}E\\ \mathcal{D}_4\end{array}}\,\vee\mathrm{E}-1,2$$

$\exists R$-reductions: If R is an E-rule with C as its major (left) premise and $(\mathcal{D}_3)$ as the deduction(s) of its minor premises, if any, then

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ (\exists x)A(x)\end{array}\quad \begin{array}{c}1\\ [A(c)]\\ \mathcal{D}_2(c)\\ C\end{array}}{C}\,\exists\mathrm{E}-1 \qquad (\mathcal{D}_3)}{\begin{array}{c}E\\ \mathcal{D}_4\end{array}}\,R$$

reduces to

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ (\exists x)A(x)\end{array}\quad \dfrac{\begin{array}{c}1\\ [A(c)]\\ \mathcal{D}_2(x)\\ C\end{array}\quad (\mathcal{D}_3)}{E}\,R}{\begin{array}{c}E\\ \mathcal{D}_4\end{array}}\,\exists\mathrm{E}-1$$

To these reduction steps, we add one for rule Ds:

DsR-reduction steps: If R is an E-rule with C as its major (left) premise and

$(\mathcal{D}_0)$ as the deduction(s) of its minor premises, if any, then

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}1\\ {[C_1]}\\ \mathcal{D}_1\\ C\end{array}\quad \begin{array}{c}2\\ {[C_2]}\\ \mathcal{D}_2\\ C\end{array}\quad \ldots \quad \begin{array}{c}n\\ {[C_n]}\\ \mathcal{D}_n\\ C\end{array}}{C}\,\text{Ds}-1,2,\ldots,n \qquad (\mathcal{D}_0)}{E}\,R\\ \mathcal{D}_{n+1}\end{array}$$

reduces to

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}1\\ {[C_1]}\\ \mathcal{D}_1\\ C\end{array}\quad (\mathcal{D}_0)}{E}\,R \qquad \dfrac{\begin{array}{c}2\\ {[C_2]}\\ \mathcal{D}_2\\ C\end{array}\quad (\mathcal{D}_0)}{E}\,R \qquad \ldots \qquad \dfrac{\begin{array}{c}n\\ {[C_n]}\\ \mathcal{D}_n\\ C\end{array}\quad (\mathcal{D}_0)}{E}\,R}{E}\,\text{Ds}-1,2,\ldots,n\\ \mathcal{D}_{n+1}\end{array}$$

Theorem 2 *Every deduction in this disjunctive intermediate logic can be reduced to a normalized deduction.*

Proof Similar to the proof by Prawitz [8, Chapter IV Theorem 1] of the normalization of minimal and intuitionistic predicate logic. Extend the definition of segment so that if a premise of rule Ds in in a segment, then so is the conclusion. Then Prawitz' procedure for removing maximum segments works for this system. ■

Remark 1 I conjecture that it is possible to prove strong normalization (that every sequence of reduction steps terminates), but I have not tried to find a proof.

3 A Gentzen L-formulation

Let us recall that the Gentzen formulation LM of minimal logic is defined as follows: the axiom scheme is

Ax $A \vdash A$.

The structural rules are as follows:

$$*\mathrm{C}\quad \frac{\Gamma_1, B, A, \Gamma_2 \ \vdash\ C}{\Gamma_1, A, B, \Gamma_2 \ \vdash\ C}$$

$$*\mathrm{K}\quad \frac{\Gamma \ \vdash\ C}{\Gamma, A \ \vdash\ C}$$

$$*\mathrm{W}\quad \frac{\Gamma, A, A \ \vdash\ C}{\Gamma, A \ \vdash\ C}$$

$$\mathrm{Cut}\quad \frac{\Gamma, A \ \vdash\ C \qquad \Gamma \ \vdash\ A}{\Gamma \ \vdash\ C}$$

The operational rules are:

$$*\wedge\quad \frac{\Gamma, A_i \ \vdash\ C}{\Gamma, A_1 \wedge A_2 \ \vdash\ C} \qquad\qquad \wedge*\quad \frac{\Gamma \ \vdash\ A_1 \qquad \Gamma \ \vdash\ A_2}{\Gamma \ \vdash\ A_1 \wedge A_2}$$

$$*\vee\quad \frac{\Gamma, A_1 \ \vdash\ C \qquad \Gamma, A_2 \ \vdash\ C}{\Gamma, A_1 \vee A_2 \ \vdash\ C} \qquad\qquad \vee*\quad \frac{\Gamma \ \vdash\ A_i}{\Gamma \ \vdash\ A_1 \vee A_2}$$

$$*\supset\quad \frac{\Gamma \ \vdash\ A \qquad \Gamma, B \ \vdash\ C}{\Gamma, A \supset B \ \vdash\ C} \qquad\qquad \supset*\quad \frac{\Gamma, A \ \vdash\ B}{\Gamma \ \vdash\ A \supset B}$$

$$*\forall\quad \frac{\Gamma, A(t) \ \vdash\ C}{\Gamma, (\forall x)A(x) \ \vdash\ C} \qquad\qquad \forall*\quad \frac{\Gamma \ \vdash\ A(c)}{\Gamma \ \vdash\ (\forall x)A(x)}$$

$$*\exists\quad \frac{\Gamma, A(c) \ \vdash\ C}{\Gamma, (\exists x)A(x) \ \vdash\ C} \qquad\qquad \exists*\quad \frac{\Gamma \ \vdash\ A(t)}{\Gamma \ \vdash\ (\exists x)A(x)}$$

Here, in rules $*\wedge$ and $\vee*$, $i = 1$ or $i = 2$; in rules $*\forall$ and $\exists*$, t is any term; and in rules $\forall*$ and $*\exists$, c is a variable which does not occur free in Γ (or C).

The system LJ is obtained from LM by adding the rule

$$\bot\mathrm{J}\quad \frac{\Gamma \ \vdash\ \bot}{\Gamma \ \vdash\ A}$$

A *cut-free derivation* is a derivation in which rule (cut) does not occur.

The Gentzen-style L-rule corresponding to Ds is

$$\mathrm{Dx}\quad \frac{\Gamma, C_1(\vec{A}) \ \vdash\ C \qquad \Gamma, C_2(\vec{A}) \ \vdash\ C \qquad \ldots \qquad \Gamma, C_n(\vec{A}) \ \vdash\ C}{\Gamma \ \vdash\ C}$$

Definition 2 The *system LI* is obtained from the system LJ by adding rule Dx.

Theorem 3 *If*

$$\Gamma \vdash A$$

is derivable in system LI, then there is a deduction of it in TI.

Proof An easy induction on the derivation of $\Gamma \vdash A$. ■

Theorem 4 *If there is a normal deduction of*

$$\Gamma \vdash A$$

in TI, then there is a cut-free derivation of it in LI.

Proof Similar to the proof of Prawitz [8, Appendix A, Theorem]. An extra case for Ds is needed, where Ds is neither an I-rule nor an E-rule; this case is easy using Dx. ■

Theorem 5 *If*

$$\Gamma \vdash A$$

can be derived in LI, then there is a cut-free derivation of it.

Proof Theorems 2, 3, and 4. ■

Remark 2 Curry [4, p. 262] uses as his rule Nx a special case of Dx. However, he does not use the corresponding rule for his rule Px, which he takes instead in the form

$$\frac{\Gamma, A \supset B \vdash A}{\Gamma \vdash A}$$

Remark 3 There are properties that follow from normalization for minimal and intuitionistic logic which do not hold for TI or LI. These include the separation property (which says that only if a connective or quantifier appears in an undischarged assumption or in the conclusion are its rules used in the deduction) and the subformula property (which says that every formula occurring in a deduction occurs in one of the undischarged assumptions or in the conclusion). It is easy to see that rule Ds (or Dx) may result in the discharge (disappearance from the cut-free deduction) of a connective or quantifier or of a subformula that does not occur in another undischarged assumption or in the conclusion. Furthermore, it is easy to see that rule Dx is a special case of (cut). This suggests that the normalization procedure given in this paper is not complete.

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