More Thoughts on Teaching Elementary Mathematics

Jonathan P. Seldin
Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, Alberta, Canada
jonathan.seldin@uleth.ca
http://www.cs.uleth.ca/∼seldin

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This paper is a continuation of the paper I presented last year, [7].

In that paper, I referred to a talk given in 2005 by Keith Devlin [3], in which he divided mathematics into two kinds:

- Elementary mathematics: Definitions can be understood when given.
- Formal mathematics: Definitions must be used before they can be understood.

Devlin suggested that formal mathematics is very difficult for some people to learn, and went on to suggest that some people may never be able to learn formal mathematics.

I suggested in [7] that "Formal" applies to the presentation, not the subject matter. After all, definitions which are formal to most of us are not formal to those who first thought of them. This suggests that the difficulty that many people have in learning from a formal presentation is an educational problem that it should be possible to overcome. The obvious way to try to solve the problem is to:

- Make presentations elementary before the transition course to advanced mathematics.
- Design the transition course so that students who finish it can learn formally presented mathematics.

Since many students who have taken this transition course have still not learned how to learn formally presented mathematics, it may be necessary to apply these ideas also to some courses that students take after this transition course.

Sherry Mantyka, of Memorial University, who runs a program of remedial mathematics for students who are weak in the subject, referred in [4], to three stages of mathematical learning:

1. Present the rule to be learned, and, in the process explain why the rule makes sense.

2. Practice the rule with many examples.

3. Use the rule in problem solving.

The importance of the second step is that our working memory is fairly small, and we cannot do with our brains what we do with our computers, namely, take them to a dealer for a memory upgrade. The point of practicing with many examples is to make the use of the rule automatic, which means transferring its use to another part of the brain, thus freeing up our working memory for other tasks. This is true in learning to play a sport or learning to play a musical instrument as well as the learning of mathematics.

The ideas of this paper concern the first of these steps.

Last year, in [7], I suggested that one place to apply the ideas of this paper is to a first, third-year course in analysis. I was able to try this in the fall of 2006 because I was teaching that course here at the University of Lethbridge. Furthermore, since the Department of Mathematics and Computer Science at this university had already decided to introduce a fourth-year course in analysis, I felt able to take a risk that might result in some topics not being covered, since students could cover those topics in a later course. I proposed to run the course as a critique of calculus and use it as a vehicle to introduce the theory of analysis. Here is the catalog description:

> Rigorous treatment of the notions of calculus of a single variable, emphasizing epsilon-delta proofs. Completeness of the real numbers. Upper and lower limits. Continuity. Differentiability. Riemann integrability.

Here is the proposed course outline:

> 1. Introduction to a critique of calculus. In evaluation of derivative of $f(x)$ at $x = a$, we do manipulations that depend on x not being a and then substitute a for x. Why can we do this? Berkeley's [1]. How can we

justify using calculus? What constitutes a proof in mathematics? (Ideas from outline for transition course if students have not had it.)

2. Ancient Greek approach to numbers and magnitudes. Zeno's paradoxes. Some pre-Euclidean proofs. Knorr's proof that the side and diagonal of a square are incommensurable. Discussion of this and what it could mean to measure a length *exactly*.

3. Idea behind limits. Preliminary theory of limits. Have students list limit theorems from their calculus text (or give them such a list) and have them determine which ones need to be taken as axioms from which the others can be proved. Show that this is incomplete. Then use my paper [6] to get $\epsilon - N$ definition, and from this get $\epsilon - \delta$.

4. What must be true of quantities or numbers for all this to work? Why rationals are not enough. How can we get reals? Evolution from ancient Greek ideas to modern ideas. Construction of number systems.

5. Countable and uncountable sets. Start with Galileo's observation that positive integers can be paired with the positive perfect squares. Additional examples. Uncountability of reals, and proof that set of real valued functions of real numbers has higher cardinality than reals. Introduce language of set theory from this, and also discuss functions and their properties.

6. Metric topological properties of real numbers. More on limits. Continuity. Sequences, series, and convergence.

7. Derivatives. Use of limits in defining. The derivative is a function of a function.

8. Riemann integrability. Properties of lubs and glbs needed to prove properties.

9. If time, measure theory and Lebesgue integral. Start with question of what sets of points of discontinuity a function can have and still have an integral, based on Fourier series.

When I taught this course last fall. I was unable to cover all the outline: I had to leave out 5, most of 6, 7, and 9 above. I also learned some things about the following items from the outline:

2. It seems useful to cover the work of Theodorus and Theaetetus on incommensurable magnitudes. This course should be partly a study of the kinds of arguments used in the past in dealing with the subject. It might also be useful to spend some time on the theory of even and odd numbers, which is considered to be the oldest deductive theory of which we have a record in Book VII

of Euclid's *Elements*. Students could be assigned exercises to construct pebble diagrams.

3. My original idea for the first theory of limits was to list limit properties familiar from calculus and ask students to come to the board and prove some from others. But some students felt discriminated against because they were too shy, and this also took up too much time. I do not plan to try this again. Instead, I will simply preseent the axioms and give exercises for some of the theorems.

   Getting the $\epsilon - N$ definition of a limit from theorems of Euclid and Archimedes: since Archimedes (in his work on the measurement of a circle) used the construction from Euclid (due to Eudoxus) from the theorem that circles are to each other as the squares of their diameters, and since Archimedes assumed that the construction was well known, the theorem from Euclid should be covered first. Also, there are places where students need annotations to the proofs in Euclid and Archimedes, and there are places at which they need sentences written as formulas.

   Once the $\epsilon - N$ and epsilon-delta definitions are given, students need some "how to" instructions for constructing proofs of this kind.

4. The Axiom of Completeness for the real numbers can be justified from the $\epsilon - N$ definition as follows: consider an increasing sequence bounded from above. Intuitively, this sequence must have a limit. But assuming that the limit is anything other than the least upper bound of the elements of the sequence negates the $\epsilon - N$ definition.

   For suppose a_n is an increasing sequence bounded from above.

   1. Suppose that a is a number for which $a_n \leq a$ for all n is false. Then for some N, $a < a_N$. Since the sequence is increasing, $a < a_N < a_m$ for all $m > N$. Let $\epsilon = a_N - a > 0$. Then for all $m > N$, $a_m > a_N > a$, so $|a_m - a| > \epsilon$ for all $m > N$. Hence, by the definition, a is not the limit of the sequence.
   2. Suppose that there is a number $b < a$ such that $b \geq a_n$ for all n. Let $\epsilon = a - b > 0$. Then, since $a > b \geq a_n$ for all n, $|a - a_n| \geq \epsilon$ for all n, and so a is not the limit of the sequence.

   It follows that the limit of the sequence must be the least upper bound.

To guarantee that any increasing sequence bounded from above has a limit, it seems necessary to have the Completeness Axiom for the reals.

This can be shown without using the phrase "least upper bound". It can be presented without a formal presentation. The phrase "least upper bound", or "supremum", can be defined after this presentation, when a name seems desirable.

It is also easy to show from this that the rational numbers are not enough. Dedekind, in [2], gives a formula which can be used to generate a sequence of rational numbers approaching $\sqrt{n}$ for any n either from below or from above. Another formula for generating a sequence approaching $\sqrt{2}$ from below is given by Perron in [5]. These make it easy to show that there is no least upper bound for a sequence of rational numbers approaching $\sqrt{2}$ from below.

Because of the course catalog description, I felt a need to include lim sup and lim inf in the course. Last fall, I could not find a good non-formal way to do this. On thinking about it, I think I should start by talking about tails of sequences, emphasizing that if the first N terms of a sequence are replaced, the limit is unchanged, even if N is very large. This clearly means that from N on, the tails are unchanged. It should be easy to discuss properties of tails without using a formal presentation, and show that for a bounded sequence, the tails are bounded and hence have inf's and sup's. I hope to find a way to lead into a discussion of lim sup's and lim inf's without using a formal presentation.

References

[1] George Berkeley. The Analyst: or A Discourse Addressed to an Infidel Mathematician. Wherein it is examined whether the object, principles, and inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and points of Faith. First printed in 1734. In A. A. Luce and T. E. Jessop, editors, *The Works of George Berkeley Bishop of Cloyne*, volume 4, pages 53–102. Nelson, London, 1951. Full text, edited by David R. Wilkins, available on line at `http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.html`.

[2] Richard Dedekind. *Was sind und was sollen die Zahen?* Friedr. Vieweg & Sohn, Braunschweig, 10th edition, 1965. 1st edition, 1887.

[3] K. Devlin. How much mathematics can be for all? Plenary Talk, Summer 2005 Meeting of Canadian Mathematical Society with Canadian Society for History

and Philosophy of Mathematics, University of Waterloo, Waterloo, Ontario, June 4–6, 2005.

[4] S. Mantyka. The math plague. Coast-to-Coast Seminar, December 2006.

[5] O. Perron. *Irrationalzahlen*. Chelsea, 1951. Translated from the second German edition, 1939 (date found from an advertisement). The preface to the first edition is dated December, 1920.

[6] J. P. Seldin. From exhaustion to modern limit theory. In Francine F. Abeles, Victor J. Katz, and Robert S. D. Thomas, editors, *Canadian Society for History and Philosophy of Mathematics. Proceedings of the Sixteenth Annual Meeting, Victoria, British Columbia, May 31–June 1, 1990*, pages 120–136, 1990. Available on line from `http://www.cs.uleth.ca/~seldin` under "Publications.".

[7] J. P. Seldin. Thoughts on teaching elementary mathematics. In *Proceedings of the Canadian Society for the History and Philosophy of Mathematics, Volume 19, Thirty-First Annual Meeting, York University, Toronto, May 28–30, 2006*, pages 242–252. Canadian Society for the History and Philosophy of Mathematics, 2007. Available on line from `http://www.cs.uleth.ca/~seldin` under "Publications.".