Manipulating Proofs*

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Abstract

The purpose of this paper is to explain an approach to formal logic which is concerned with *provability* rather than with *truth* alone, as in the traditional approach. The traditional approach to propositional logic will probably be familiar to programmers, since it has been incorporated into most programming languages and spreadsheets. The approach explained in this paper has, however, become important in computer science in recent years. It is characterized by taking proofs as objects which can be manipulated.

The idea of treating proofs as mathematical objects to be manipulated goes back to Hilbert, who, in response to the attacks on classical mathematics from the intuitionists, proposed to ground classical mathematics by treating proofs as arrays of meaningless symbols (using a formalized language) and proving by unquestionably valid combinatorial techniques that no proof could end in a formula with the form of a contradiction. Gödel's Second Theorem, which shows that it is impossible to prove the consistency of any system (strong enough to be interesting) by means of proof which can be expressed

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within the system, is usually interpreted to mean that Hilbert's idea will not work in the way that he intended it to. However, some mathematicians and philosophers have disputed this interpretation, and, over the years, a number of important mathematical systems have been proved consistent in this way, including the elementary theory of numbers and elementary classical analysis.

This paper is an introduction to techniques of this kind due to Gerhard Gentzen [2] and Dag Prawitz [3], with emphasis on propositional logic. The ideas have become important in computer science in recent years.

We begin in §1 with a discussion of propositional logic as given by truth tables. In §2, we will look at a natural deduction formulation of this logic, system TM. In §3, we will look at normalization for pure implication formulas. This is a method for proving consistency and other results that is due to Dag Prawitz [3]. In §4, we will look at normalization and its consequences for the full natural deduction system introduced in §2, and we will see that that system is not complete in that not all truth-table tautologies are provable. We will also define two stronger systems, TJ and TK. In §5, we will look L-formulations of TM and TJ; these are systems due to Gentzen [2] which are useful in proof searches. In §6, we will look at an L-formulation of TK, and we will use this system that every tautology can be proved in TK.

The first part of this paper, §1 – §4, was presented at a seminar talk at the Department of Mathematics at the University of Wollongong in Australia on 31 July 1986, and has been presented in a number of other talks since.

1 Propositional logic with truth tables

Until further notice, we shall deal with *formulas* built up from *atomic formulas* by means of connectives $\wedge$ (and), $\vee$ (or), and $\supset$ (implies). One of the atomic formulas will be $\perp$ (absurdity), and negation will be defined using absurdity by $\neg A \equiv A \supset \perp$.

In traditional logic, formulas are assigned *truth values* by means of *truth tables*. There are two truth values: T (true) and F (false). Absurdity, $\perp$, is always assigned the value F. For any given assignment of truth values to the other atomic formulas, truth values are assigned to compound formulas by the following table:

A	B	$A \wedge B$	$A \vee B$	$A \supset B$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

This gives us the following derived truth table for negation:

A	$\neg A$
T	F
F	T

A formula which has a truth table in which all of the entries for it are T is called a *tautology*. An example of a tautology is $\neg A \vee A$, which has the following truth table:

A	$\neg A$	$\neg A \vee A$
T	F	T
F	T	T

This traditional approach to propositional logic is concerned with *truth* but not with *provability*. It has been incorporated into programming languages (and spreadsheets) in the *boolean data type*. (But in this incorporation, $\supset$ is usually not defined, and our derived table for negation is taken as primitive. In such a logic, $A \supset B$ can be defined to be $\neg A \vee B$, which has the same truth table.)

2 Logic with proofs as trees: natural deduction

One of the most common ways of taking proofs as objects to be manipulated is to write proofs as *tree diagrams*. We will do this for a system of *natural deduction*. In this system there are no axioms, and the rules come in pairs, one for introducing a given connective and the other for eliminating it.

For example, a conjunction can be introduced by the following rule:

Rule for Conjunction Introduction ($\wedge$I) From A and B to infer $A \wedge B$.

This rule is written in the form of a tree diagram as follows:

$$\frac{A \quad B}{A \wedge B.}$$

Here, the premises are written above the line and the conclusion below it. If, as these rules are combined to form larger deductions, we want to indicate the rule used, we can write it to the right of the line:

$$\frac{A \quad B}{A \wedge B.}\ \wedge\mathrm{I}$$

When a tree diagram of a deduction is constructed in this way, the idea is that the formulas at the tops of the branches are the *assumptions* on which the conclusion depends.

For eliminating conjunction, we have the following rules:

Rule for Conjunction Elimination (∧E) From $A \wedge B$ to infer either A or B.

In tree form, we get the following:

$$\frac{A \wedge B}{A,} \qquad\qquad \frac{A \wedge B}{B.}$$

For disjunction, we also have two rules for its introduction:

Rules for Disjunction Introduction (∨I) From either A or B to infer $A \vee B$.

In tree form,this is

$$\frac{A}{A \vee B,} \qquad\qquad \frac{B}{A \vee B.}$$

The rule for eliminating disjunction is more complicated:

Rule for Disjunction Elimination (∨E) From $A \vee B$, a deduction of C from A (and the other assumptions), and a deduction of C from B (and the other assumptions), to infer C.

This is *proof by cases*. The complication is that the second and third premises are not formulas but entire deductions. Furthermore, the conclusion C of the first of these deductions depends on one more assumption, A, than the conclusion C of the inference, and the conclusion C of the second of these deductions depends on one more assumption, B, than does the conclusion C of the inference. This is indicated in the tree diagram by putting square brackets around these extra assumptions as follows:

$$\frac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C.}$$

The assumptions in brackets are said to be *discharged* by the inference, and this is a sign that the premises are, indeed, entire deductions. In a diagram of a multi-rule deduction, when it is desired to indicate the inferences at which formulas are discharged, numbers are used as follows:

$$\frac{A \vee B \quad \begin{matrix}1 \\ [A] \\ C\end{matrix} \quad \begin{matrix}2 \\ [B] \\ C\end{matrix}}{C.} \; (\vee\mathrm{I} - 1 - 2)$$

A simpler form of discharging of assumptions occurs in the rule for introducing implication:

Rule for Implication Introduction ($\supset$I) From a deduction of B from A (and other assumptions), to infer $A \supset B$.

This is the rule of *conditional proof*. In tree form, we have

$$\frac{\begin{matrix}[A] \\ B\end{matrix}}{A \supset B.}$$

The rule for eliminating $\supset$ is *modus ponens*:

Rule for Implication Elimination ($\supset$ E) From $A \supset B$ and A to infer B.

In tree form, this is

$$\frac{A \supset B \quad A}{B.}$$

By using the definition of $\neg A$ as $A \supset \perp$, it is possible to derive the following introduction and elimination rules for it:

Rule for Negation Introduction ($\neg$I) From a deduction of $\perp$ from A (and other assumptions) to infer $\neg A$.

This is a form of the rule of *indirect proof*. In tree form, it is

$$\frac{\begin{array}{c}[A]\\ \perp\end{array}}{\neg A.}$$

Rule for Negation Elimination ($\neg$E) From $\neg A$ and A to infer $\perp$.

In tree form:

$$\frac{\neg A \quad A}{\perp.}$$

This shows that $\perp$ stands for a general contradiction.

Definition 1 The *System* TM has formulas built up from atomic formulas (one of which is $\perp$) by means of the binary connectives $\wedge$, $\vee$, and $\supset$. It has no axioms. Its rules are ($\wedge$I), ($\wedge$E), ($\vee$I), ($\vee$E), ($\supset$I), ($\supset$E). If Γ is a set of formulas, then we say that A can be *deduced* from Γ when there is a deduction in tree form built up using these six rules in which the formula at the bottom is A and every assumption (formula at the top of a branch) which is not discharged is in Γ. In this case, we write

$$\Gamma \vdash_{\mathrm{TM}} A,$$

or, if the context makes the system clear,

$$\Gamma \vdash A.$$

A *proof* is a deduction in which there are no undischarged assumptions. To say that there is a proof of A, we write

$$\vdash_{\mathrm{TM}} A,$$

or, if the context makes the system clear,

$$\vdash A.$$

This may perhaps be made a bit clearer with some examples:

Example 1 Proof of $\vdash A \supset A$:

$$\dfrac{\overset{1}{[A]}}{A \supset A.} \supset \mathrm{I} - 1$$

A formula may be discharged even if it does not actually occur at the top of a branch:

Example 2 Proof of $\vdash A \supset (A \supset B)$:

$$\dfrac{\dfrac{\overset{1}{[A]}}{B \supset A}\,(\supset \mathrm{I} - \mathrm{v})}{A \supset (B \supset A).}\,(\supset \mathrm{I} - 1)$$

Here, the assumption B which is discharged at the first inference by $(\supset \mathrm{I})$ does not actually occur at the top of a branch, and this is indicated by the label "$(\supset \mathrm{I} - \mathrm{v})$" (the "v" stands for "vacuous").

More than one occurrence of an assumption can be discharged at one inference:

Example 3 Proof of $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{[A \supset (B \supset C)]} \quad \overset{1}{[A]}}{B \supset C}\,(\supset \mathrm{E}) \quad \dfrac{\overset{2}{[A \supset B]} \quad \overset{1}{[A]}}{B}\,(\supset \mathrm{E})}{C}\,(\supset \mathrm{E})}{A \supset C}\,(\supset \mathrm{I} - 1)}{(A \supset B) \supset (A \supset C)}\,(\supset \mathrm{I} - 2)}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)).}\,(\supset \mathrm{I} - 3)$$

Here, the two occurrences of assumption number 1 are discharged at the first inference by $(\supset \mathrm{I})$.

Sometimes it is necessary to derive a formula more than once in order to be able to discharge enough assumptions:

Example 4 Proof of $\vdash (A \supset (A \supset B)) \supset (A \supset B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{[A \supset (A \supset B)]} \quad \overset{1}{[A]}}{A \supset B}\,(\supset \mathrm{E}) \qquad \overset{1}{[A]}}{B}\,(\supset \mathrm{E})}{A \supset B}\,(\supset \mathrm{I}-1)}{(A \supset (A \supset B)) \supset (A \supset B).}\,(\supset \mathrm{I}-2)$$

Example 5 Proof of $A \supset B, B \supset C \vdash A \supset C$:

$$\dfrac{\dfrac{B \supset C \qquad \dfrac{A \supset B \quad \overset{1}{[A]}}{B}\,(\supset \mathrm{E})}{C}\,(\supset \mathrm{E})}{A \supset C.}\,(\supset \mathrm{I}-1)$$

3 Reductions of deductions with only implication formulas

In this section, we consider formulas without $\wedge$ and $\vee$, which means that here our formulas are built from the atomic formulas using only $\supset$. We shall look at manipulations on deductions using these formulas from which some important metatheorems (including consistency) can be proved.

$\supset$-reduction step A $\supset$-reduction step is the replacement of a part of a deduction of the form

$$\dfrac{\dfrac{\begin{matrix}\overset{1}{[A]}\\ \mathcal{D}_1\\ B\end{matrix}}{A \supset B}\,(\supset \mathrm{I}-1) \qquad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{\begin{matrix}B\\ \mathcal{D}_3\end{matrix}}\,(\supset \mathrm{E})$$

by

$$\begin{matrix}\mathcal{D}_2\\ A\\ \mathcal{D}_1\\ B\\ \mathcal{D}_3.\end{matrix}$$

(Note that the number of occurrences of $\mathcal{D}_2$ needed in the reduced deduction is the same as the number of occurrences of the assumption discharged at the indicated inference by ($\supset$I) in the first deduction.) The indicated formula $A \supset B$ in the first deduction is called the *cut formula* of the reduction step. A *reduction* is a (possibly empty) sequence of reduction steps.

Example 6 Consider the deduction

$$
\begin{prooftree}
\AxiomC{$\overset{2}{[A \supset (A \supset A)]}$}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{$A \supset A$}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{A}
\RightLabel{$(\supset \mathrm{I}-1)$}
\UnaryInfC{$A \supset A$}
\RightLabel{$(\supset \mathrm{I}-2)$}
\UnaryInfC{$(A \supset (A \supset A)) \supset (A \supset A)$}
\AxiomC{$\overset{3}{[A]}$}
\RightLabel{$(\supset \mathrm{I}-\mathrm{v})$}
\UnaryInfC{$A \supset A$}
\RightLabel{$(\supset \mathrm{I}-3)$}
\UnaryInfC{$A \supset (A \supset A)$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{$A \supset A.$}
\end{prooftree}
$$

It has a cut formula (the major [or left] premise of the last inference). Thus, a reduction step leads to

$$
\begin{prooftree}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\supset \mathrm{I}-\mathrm{v})$}
\UnaryInfC{$A \supset A$}
\RightLabel{$(\supset \mathrm{I}-1)$}
\UnaryInfC{$A \supset (A \supset A)$}
\AxiomC{$\overset{2}{[A]}$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{$A \supset A$}
\AxiomC{$\overset{2}{[A]}$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{A}
\RightLabel{$(\supset \mathrm{I}-2)$}
\UnaryInfC{$A \supset A.$}
\end{prooftree}
$$

There is also a cut formula here: $A \supset (A \supset A)$. (Note that this cut formula was created by the previous reduction step.) Another reduction step leads to

$$
\begin{prooftree}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\supset \mathrm{I}-\mathrm{v})$}
\UnaryInfC{$A \supset A$}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\supset \mathrm{E})$}
\BinaryInfC{A}
\RightLabel{$(\supset \mathrm{I}-1)$}
\UnaryInfC{$A \supset A.$}
\end{prooftree}
$$

Here, again, there is a cut formula (the first $A \supset A$), and this reduces to Example 1.

Definition 2 A *normal deduction* is one which cannot be further reduced; i.e., a deduction in which there are no cut formulas.

In a normal deduction, no I-rule precedes an E-rule in any branch, where a branch starts at the top of a deduction and ends with a minor (right) premise for an inference by ($\supset$E).

Theorem 1 (Normalization) *Every deduction can be reduced to a normal deduction with the same undischarged assumptions and the same conclusion.*

Proof Define the *rank* of a formula to be the number of connectives occurring in it. The proof is by a double induction, first on the maximum rank of any cut formula in the given deduction, and second by the number of cut formulas with that maximum rank. If there are any cut formulas in the deduction, reduce a cut formula with maximum rank such that there is no other cut formula with this maximum rank above this cut formula or above the minor premise associated with it. Any new cut formulas introduced by the reduction will have lower rank, so either the number of cut formulas of maximum rank is reduced by one or, if that number was already one before the reduction, the maximum rank of a cut formula in the deduction is reduced by one. ■

Theorem 2 (Consistency) *There is no proof (deduction without undischarged assumptions) whose conclusion is an atomic formula.*

Proof If there were such a proof, there would be such a deduction which is normal. Now the main (left) branch of a normal deduction which ends in an atomic formula cannot have any inferences by ($\supset$I); for these inferences would have to occur at the end of the branch, and this would mean that the last inference of the deduction is by ($\supset$I), contradicting the hypothesis that the conclusion is an atomic formula. But since ($\supset$I) is the only rule we have here that discharges an assumption, the formula at the top of the main branch is not discharged, contradicting the assumption that the deduction is a proof. ■

This theorem gives us the consistency of the system. In particular, since $\perp$ is an atomic formula, we have the following corollary:

Corollary 2.1 *Not* $\vdash \perp$.

The fact that in every branch of a normal deduction all inferences by ($\supset$E) precede all inferences by ($\supset$I) gives us the following result:

Theorem 3 (Subformula property) *Each formula occurring in a normal deduction is a subformula of the conclusion or of one of the undischarged assumptions.*

This theorem is useful in searching for proofs. Later in the paper, we shall see more on proof searches.

4 Reductions of all deductions

Let us now extend the results of the previous section to all deductions. To begin, we need reduction steps for $\wedge$ and $\vee$.

$\wedge$-reduction step A $\wedge$-reduction step is a replacement of a part of a deduction of the form

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_2 \\ \dfrac{\dfrac{A_1 \quad A_2}{A_1 \wedge A_2}\ \wedge\mathrm{I})}{A_i}\ (\wedge\mathrm{E}) \\ \mathcal{D}_3 \end{array}$$

(where $i = 1$ or $i = 2$) by

$$\begin{array}{c} \mathcal{D}_i \\ A_i \\ \mathcal{D}_3. \end{array}$$

$\vee$-reduction step A $\vee$-reduction step is a replacement of a part of a deduction of the form

$$\begin{array}{c} \dfrac{\begin{array}{c}\mathcal{D}_0 \\ \dfrac{A_i}{A_1 \vee A_2}\ (\vee\mathrm{I})\end{array} \quad \begin{array}{c}1 \\ [A_1] \\ \mathcal{D}_1 \\ C\end{array} \quad \begin{array}{c}2 \\ [A_2] \\ \mathcal{D}_2 \\ C\end{array}}{C}\ (\vee\mathrm{E} - 1 - 2) \\ \mathcal{D}_3 \end{array}$$

(where $i = 1$ or $i = 2$) by

$$\begin{array}{c} \mathcal{D}_0 \\ A_i \\ D_i \\ C \\ \mathcal{D}_3. \end{array}$$

This is the step if both discharged assumptions actually occur. If at least one does not occur, we have the following *immediate simplifications*: replacement of a deduction of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ A_i\end{matrix}}{A_1 \vee A_2}\,(\vee\mathrm{I}) \qquad \begin{matrix}\mathcal{D}_1\\ C\end{matrix} \qquad \begin{matrix}2\\ [A_2]\\ \mathcal{D}_2\\ C\end{matrix}}{\begin{matrix}C\\ \mathcal{D}_3\end{matrix}}\,(\vee\mathrm{E}-1-2)$$

by

$$\begin{matrix}\mathcal{D}_1\\ C\\ \mathcal{D}_3\end{matrix}$$

or of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ A_i\end{matrix}}{A_1 \vee A_2}\,(\vee\mathrm{I}) \qquad \begin{matrix}1\\ [A_1]\\ \mathcal{D}_1\\ C\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ C\end{matrix}}{\begin{matrix}C\\ \mathcal{D}_3\end{matrix}}\,(\vee\mathrm{E}-1-2)$$

by

$$\begin{matrix}\mathcal{D}_2\\ C\\ \mathcal{D}_3.\end{matrix}$$

We would like to reduce deductions as before by eliminating *cut formulas*, where a cut formula is the conclusion of an I-rule which is also the major (left) premise of an E-rule. If we only had to consider $\wedge$-reductions (along with $\supset$-reductions), there would be no problem: each $\wedge$-reduction step actually reduces the size of the deduction, so we would only have to reduce a cut formula of maximum rank with no cut formulas of maximum rank above it (or its minor premise, if any), and Theorem 1 would be proved as before. But with $\vee$ in the picture, this is not enough. For consider the following deduction of $(A \supset B) \vee (A \supset C), A \vdash B \vee C$:

$$\dfrac{\dfrac{(A \supset B) \vee (A \supset C) \qquad \dfrac{\dfrac{\dfrac{\overset{3}{[A \supset B]} \quad \overset{1}{[A]}}{B}\,(\supset \mathrm{E})}{B \vee C}\,(\vee\mathrm{I})}{A \supset B \vee C}\,(\supset \mathrm{I}-1) \qquad \dfrac{\dfrac{\dfrac{\overset{4}{[A \supset C]} \quad \overset{2}{[A]}}{C}\,(\supset \mathrm{E})}{B \vee C}\,(\vee\mathrm{I})}{A \supset B \vee C}\,(\supset \mathrm{I}-2)}{A \supset B \vee C}\,(\vee\mathrm{E}-3-4) \qquad A}{B \vee C.}\,(\supset \mathrm{E})$$

This deduction contains no cut formulas, but we would like to reduce it to

$$\dfrac{(A \supset B) \vee (A \supset C) \qquad \dfrac{\dfrac{\overset{1}{[A \supset B]} \quad A}{B}\,(\supset \mathrm{E})}{B \vee C}\,(\vee\mathrm{I}) \qquad \dfrac{\dfrac{\overset{2}{[A \supset C]} \quad A}{C}\,(\supset \mathrm{E})}{B \vee C}\,(\vee\mathrm{I})}{B \vee C.}\,(\vee\mathrm{E}-1-2)$$

To carry out this deduction, we need to consider *segments*. In the first deduction, the segments (there are two of them) contain the formula $A \supset B \vee C$; in the second, they contain $B \vee C$. The segments in the first deduction are *maximum segments* because their first formulas are conclusions of an I-rule and their last formulas are major (left) premises for an E-rule. Here is the definition:

Definition 3 A sequence of formulas $C_1, C_2, \ldots, C_n$ is a *segment* if they are all the same formula and, for $i = 1, \ldots, n-1$, C_i is a minor premise for an inference by ($\vee$E) for which C_{i+1} is the conclusion. The segment is a *maximum segment* if C_1 is the conclusion of an I-rule and C_n is the major (left) premise of an E-rule. Note that a cut formula is a maximum segment of length 1.

To deal with maximum segments of length two or more, we use the following reduction step:

$\vee$E-reduction step A $\vee$-*reduction step* is a replacement of a part of a deduction of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ A\vee B\end{matrix}\quad\begin{matrix}1\\ [A]\\ \mathcal{D}_1\\ C\end{matrix}\quad\begin{matrix}2\\ [B]\\ \mathcal{D}_2\\ C\end{matrix}}{C}\,(\vee\mathrm{E}-1-2)\qquad(\mathcal{D}_4)}{\begin{matrix}E\\ \mathcal{D}_4\end{matrix}}\,R$$

where R is an E-rule with C as its major (left) premise and $(\mathcal{D}_4)$ as the deduction(s) of the minor premise(s), if any, by

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ A\vee B\end{matrix}\quad\dfrac{\begin{matrix}1\\ [A]\\ \mathcal{D}_1\\ C\end{matrix}\quad(\mathcal{D}_4)}{E}\,R\quad\dfrac{\begin{matrix}2\\ [B]\\ \mathcal{D}_2\\ C\end{matrix}\quad(\mathcal{D}_4)}{E}\,R}{\begin{matrix}E\\ \mathcal{D}_4.\end{matrix}}\,(\vee\mathrm{E}-1-2)$$

We can now define normal deductions for TM:

Definition 4 A *normal deduction* is one which cannot be further reduced; i.e., a deduction in which there are no maximum segments.

Theorem 4 (Normalization) *Every deduction in* TM *can be reduced to a normal deduction with the same undischarged assumptions and the same conclusion.*

Proof This is a triple induction, first on the maximum rank of any maximum segment, second on the maximum length of any maximum segment of maximum rank, and finally on the number of maximum segments with that maximum rank and maximum length. Reduction steps are applied in the following order: first, $\vee$E-reduction steps are applied to maximum segments of maximum rank and of length greater than one starting with those of maximum length. Once all maximum segments of maximum rank have length one (and so are cut formulas), reduction steps are applied to those cut formulas of maximum rank for which there is no cut formula with maximum rank above them or above any minor premise for the inference for which they are major premises. Each reduction step reduces the deduction to one that precedes it in the induction, and so the process must terminate in a normal deduction with the same undischarged assumptions and conclusion. ∎

Theorem 5 (Consistency) *There is no proof whose conclusion is an atomic formula.*

Proof Similar to the proof of Theorem 2, where we note that the left branch of a deduction ending in an atomic formula consists entirely of E-rules, and although ($\vee$E) can discharge assumptions, it cannot discharge an assumption over the left premise. ■

Corollary 5.1 *Not* $\vdash \bot$.

Proof $\bot$ is an atomic formula. ■

Theorem 6 (Subformula property) *Every formula occurring in a normal deduction is a subformula of the conclusion or of one of the undischarged assumptions.*

Proof Similar to the proof of Theorem 3. ■

Theorem 7 (Separation property) *In any normal deduction, the only rules which occur are those whose corresponding connective occurs in an undischarged assumption or in the conclusion.*

Proof Similar to the proof of Theorem 6. ■

The system TM is not complete: not all tautologies are theorems.

Theorem 8 (Disjunction property) *If* Γ *is any set of assumptions in which there is no occurrence of* $\vee$*, and if*

$$\Gamma \vdash A \vee B,$$

then

$$\Gamma \vdash A \qquad \textit{or} \qquad \Gamma \vdash B.$$

Proof Let $\mathcal{D}$ be a normal deduction of $\Gamma \vdash A \vee B$. Since $\vee$ does not occur in any undischarged assumption, the conclusion cannot be the subformula of any assumption, so $\mathcal{D}$ cannot have a left branch consisting entirely of E-rules. Hence, the last inference of $\mathcal{D}$ must be an I-rule. The only possible such rule is ($\vee$I), and if that last inference is removed we have a deduction which gives us the conclusion of the theorem. ■

Corollary 8.1 *There is a formula A for which not $\vdash \neg A \vee A$.*

Proof Let A be any atomic formula distinct from $\perp$. By Theorem 5, not $\vdash A$. Now suppose there is a normal deduction of $\vdash \neg A$. Then the last inference is by ($\supset$I), and the premise is a normal deduction of $A \vdash \perp$. By Theorem 6, every formula of this deduction must be a subformula of A or of $\perp$, and since both are atomic, this means that no other formula can occur in the deduction. Such a deduction is impossible. ∎

The system TM can be strengthened so that all tautologies become provable, and it turns out that adding a single rule is sufficient. The rule is as follows:

Rule ($\perp_{\mathrm{C}}$)

$$\frac{\begin{array}{c}[\neg A]\\ \perp\end{array}}{A.}$$

Definition 5 The *system* TK is defined by adding to the system TM the rule ($\perp_{\mathrm{C}}$). That A can be deduced in TK from Γ will be indicated by

$$\Gamma \vdash_{\mathrm{TK}} A$$

when it is necessary to indicate the system.

A proof that every tautology is a theorem of TK will be given later. For now, here is a proof that $\vdash_{\mathrm{TK}} \neg A \vee A$:

$$\dfrac{\dfrac{\overset{2}{[\neg(\neg A \vee A)]} \qquad \dfrac{\dfrac{\dfrac{\overset{2}{[\neg(\neg A \vee A)]} \qquad \dfrac{\overset{1}{[A]}}{\neg A \vee A}\,(\vee\mathrm{I})}{\perp}\,(\neg\mathrm{E})}{\neg A}\,(\neg\mathrm{I}-1)}{\neg A \vee A}\,(\vee\mathrm{I})}{\perp}\,(\neg\mathrm{E})}{\neg A \vee A.}\,(\perp_{\mathrm{C}}-2)$$

Remark 1 Note that in system TM, it is not true that any formula follows from a contradiction. There is a logic which enjoys many of the properties of

TM but in which any formula follows from a contradiction, it is the system TJ. It is formed by adding to system TM the rule $(\perp_J)$:

$$\frac{\perp}{A.}$$

This is a special case of rule $(\perp_C)$ in which the discharged assumption does not actually occur. It is sufficient to take rule $(\perp_J)$ in the special case that A is atomic, since the general case can be proved as a derived by an induction on the number of connectives and quantifiers in the conclusion. Then, because the conclusion of rule is atomic and there is no discharged assumption, adding this rule to TM does not affect normalization or any of its consequences.

Remark 2 Normalization of TK is another matter, since the conclusion of rule $(\perp_C)$ cannot be restricted to being atomic: it is not possible to break an inference by $(\perp_C)$ whose conclusion is $A \vee B$ down into inferences by the same rule whose conclusions are A or B. There are two solutions to this problem.

1. Remove $\vee$ from the list of postulated connectives and to define $A \vee B$ to be an abbreviation for $\neg A \supset B$ (which has the same truth table). (It would also be possible to define $A \wedge B$ to be an abbreviation for $\neg(A \supset \neg B)$, which has the same truth table, but this is not strictly necessary.) If the only primitive connectives are $\supset$ and $\wedge$, then it is possible to restrict $(\perp_C)$ to atomic conclusions and prove the general case by induction on the structure of the conclusion. Since the conclusion of the rule can now be assumed to be atomic, adding this rule to TM does not affect normalization or its consequences. This is the approeach of [3, Chapter III].

2. Pushing inferences by $(\perp_C)$ down to the bottom of a deduction and combining adjacent occurrences into one. This second method works even if $\vee$ is a primitive connective. The method is to transform a deduction of the form

$$\frac{\dfrac{\begin{array}{c} \overset{1}{[\neg A]} \\ \mathcal{D}_1 \\ \perp \end{array}}{A}\,(\perp_C - 1) \qquad (\mathcal{D}_2)}{B,}\; R$$

where A is a premise for an inference by rule R and $(\mathcal{D}_2)$ is (are) the deduction(s) (if any) of the other premise(s) for the inference by R, into

$$\begin{array}{c}
\dfrac{\dfrac{\overset{2}{[\neg B]} \qquad \dfrac{\overset{1}{[A]} \quad (\mathcal{D}_2)}{B}\,R}{\perp}\,(\neg\mathrm{E})}{\neg A}\,(\neg\mathrm{I}-1) \\
\mathcal{D}_1 \\
\dfrac{\perp}{B,}\,(\perp_{\mathrm{C}}-2)
\end{array}$$

provided that R does not discharge any assumptions in $\mathcal{D}_1$, and otherwise into

$$\begin{array}{c}
\dfrac{\dfrac{\overset{2}{[\neg B]} \qquad \dfrac{\overset{1}{[A]} \quad (\mathcal{D}_2)}{B}\,R}{\perp}\,(\neg\mathrm{E})}{\neg A}\,(\neg\mathrm{I}-1) \\
\mathcal{D}_1 \\
\dfrac{\dfrac{\dfrac{\perp}{A}\,(\perp_{\mathrm{C}}-\mathrm{v}) \qquad (\mathcal{D}_2)}{B}\,R}{B,}\,(\perp_{\mathrm{C}}-2)
\end{array}$$

where the discharge at the first inference by R is vaduous. This method will push down to the bottom of a deduction all inferences by $(\perp_{\mathrm{C}})$ which discharge an assumption nonvacuously, but it is necessary to be careful about the order of reductions, see [4, Theorem 4]. In what follows, we shall assume that this second method is used to normalize TK.

Remark 3 System TK can also be formulated by adding a rule to TJ. The rule is one of

$$\begin{array}{c} [\neg A] \\ \dfrac{A}{A} \end{array} \qquad \text{or} \qquad \begin{array}{c} [A \supset B] \\ \dfrac{A}{A.} \end{array}$$

On normalizing systems with these rules, see [5].

System TM is known as minimal logic, system TJ is known as *intuitionistic logic*, and system TK is known as *classical logic*.

5 System LM for proof searches in TM

So far, the proof trees have involved formulas. But if we are searching for a proof, it is more convenient to write down the premises as well as the conclusion we are trying to prove.

For example, suppose we are searching for a proof of

$$A \supset B, B \supset C \;\vdash\; A \supset C.$$

Assuming that we are working in TM, we can assume that we are searching for a normal deduction. Then the last inference must be by ($\supset$I), and so to obtain the premise we would need a normal deduction of

$$A \supset B, B \supset C, A \;\vdash\; C.$$

Since we don't know what formula C is, we must allow for the possibility of its being atomic, so we cannot assume that the last inference of this normal deduction is an I-rule; we have to assume that it is an E-rule. This would mean that the left branch consists entirely of inferences by E-rules, and by Theorem 7, these inferences must all be by ($\supset$E). It follows that the top of the left (main) branch is not discharged, and so it must be either $A \supset B$ or $B \supset C$. Suppose it is $B \supset C$; then the normal deduction looks like this:

$$\dfrac{B \supset C \qquad \begin{matrix} \mathcal{D}_1 \\ B \end{matrix}}{\begin{matrix} C \\ \mathcal{D}_2 \\ C, \end{matrix}} \ (\supset \mathrm{E})$$

where the undischarged assumptions of $\mathcal{D}_1$ and $\mathcal{D}_2$ are among $A \supset B$ and A and that C is an undischarged assumption for $\mathcal{D}_2$. This means that we need proofs of

$$A \supset B, A \;\vdash\; B \qquad\qquad \text{and} \qquad\qquad A \supset B, A, C \;\vdash\; C.$$

The second of these is trivial, so $\mathcal{D}_2$ can be the one-step deduction consisting of C. However, we need a normal deduction of the first of these. Again, we must assume that the left branch consists entirely of inferences by ($\supset$E), so the top of the left branch must be $A \supset B$ or $B \supset C$. Suppose it is $A \supset B$. By the above reasoning, we need proofs of

$$A \;\vdash\; A \qquad\qquad \text{and} \qquad\qquad A, B \;\vdash\; A.$$

Both of these are trivial, and we are led to the deduction of Example 5.

This shows that when we are searching for a deduction of

$$\Gamma, A \supset B \ \vdash \ C,$$

we may need to search for deductions of

$$\Gamma, A \supset B \ \vdash \ A \qquad\qquad \text{and} \qquad\qquad \Gamma, B \ \vdash \ C.$$

We normally will not want to repeat each time the reasoning we used above to justify this step.

For this purpose, let us consider a system in which the steps of proofs are not formulas, but statements of the form

$$\Gamma \Vdash A.$$

We will want *operational* rules in pairs, corresponding to the pairs of rules in the natural deduction systems. But we will also need some *structural* rules to cover the assumptions we have been making about natural deduction proofs. It turns out that the structural rules are easier to formulate if we now take *sequences* of assumptions instead of sets. So until further notice, assume that Γ, Δ, and Θ are *sequences* of formulas. We will refer to

$$\Gamma \Vdash A$$

as a *sequent*. (Many authors use $\Gamma \to A$ for this sequent; the notation used here is due to H. B. Curry [1]).

For the structural rules, we need one ((*C), also called *exchange*) which tells us that the order of assumptions does not matter:

$$\frac{\Gamma, A, B, \Theta \Vdash C}{\Gamma, B, A, \Theta \Vdash C.}$$

Since all this rule does is to change the order of the formulas on the left, we shall often ignore this order and this rule in what follows. We need one ((*K), also called *weakening*) which allows us to add an assumption:

$$\frac{\Gamma \Vdash C}{\Gamma, A \Vdash C.}$$

We need one ((*W), also called *contraction*) which tells us that the number of times we make an assumption does not matter:

$$\frac{\Gamma, A, A \Vdash C}{\Gamma, A \Vdash C.}$$

And finally, we need one (Cut) which tells us that $\Vdash$ is transitive:

$$\frac{\Gamma \Vdash A \quad \Gamma, A \Vdash C}{\Gamma \Vdash C.}$$

For the operational rules, those corresponding to the I-rules are fairly obvious:

Rule ($\wedge$*)

$$\frac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \wedge B,}$$

Rule ($\vee$*)

$$\frac{\Gamma \Vdash A}{\Gamma \Vdash A \vee B,} \qquad \frac{\Gamma \Vdash B}{\Gamma \Vdash A \vee B,}$$

Rule ($\supset$*)

$$\frac{\Gamma, A \Vdash B}{\Gamma \Vdash A \supset B.}$$

Corresponding to the E-rules, we assume (as above) that we are dealing with the given formula at the top of the major (left) branch. The rules are

Rule (*$\wedge$)

$$\frac{\Gamma, A \Vdash C}{\Gamma, A \wedge B \Vdash C,} \qquad \frac{\Gamma, B \Vdash C}{\Gamma, A \wedge B \Vdash C,}$$

Rule (*$\vee$)

$$\frac{\Gamma, A \Vdash C \quad \Gamma, B \Vdash C}{\Gamma, A \vee B \Vdash C,}$$

Rule (*$\supset$)

$$\frac{\Gamma \Vdash A \quad \Gamma, B \Vdash C}{\Gamma, A \supset B \Vdash C.}$$

Note that the left premise for (*$\supset$) is missing the formula $A \supset B$ on the left, which we said was needed for the proof search. We needed this formula in doing a proof search to allow for the possibility that the undischarged assumption occurs more than once in the proof we are looking for. In stating the rule, however, this formula is not needed; if this formula already occurs on the left, it will occur in Γ, and rule (*W) can then be used to combine the two occurrences into one.

This gives us pairs of operational rules which introduce formulas with the given connective on the left and on the right of the sign '$\Vdash$'. The only rule which ever removes a formula from one side or the other (without making it part of a larger formula) is (Cut). As we shall see below, rule (Cut) is redundant and can be eliminated. We are therefore interested in proofs without this rule; these proofs are called *cut-free*. Proof searches can be conducted by proceeding backwards from the conclusion using cut-free proofs.

Definition 6 The *system* LM is based on sequents of the form $\Gamma \Vdash A$, which, in turn, are based on the formulas of system TM. The *axioms* of LM are all sequents of the form

$$A \Vdash A.$$

The *structural rules* are (*C), (*K), (*W), and (Cut). The *operational rules* are (*$\wedge$), ($\wedge$*), (*$\vee$), ($\vee$*), (*$\supset$), and ($\supset$*). A proof in which rule (Cut) does not occur is called *cut-free*.

Note the derived rules for negation:

$$(*\neg) \qquad \frac{\Gamma \Vdash A \quad \Gamma, \bot \Vdash C}{\Gamma, \neg A \Vdash C} \qquad\qquad (*\neg) \qquad \frac{\Gamma, A \Vdash \bot}{\Gamma \Vdash \neg A}$$

Theorem 9 *If* Γ *is any sequence of assumptions, and if*

(1) $$\Gamma \Vdash C$$

in LM*, then*

(2) $$\Gamma \vdash_{\mathrm{TM}} C.$$

Proof By induction on the proof of (1).

Basis: (1) is an axiom. Then Γ is just C, and (2) is the one-step deduction C.

Induction step: We have cases by the last rule of the deduction of (1). The cases for the structural rules are trivial by our assumptions about deductions in TM. Even the case of (Cut) is trivial, since we automatically assume that '$\vdash$' is transitive.

Case (*$\wedge$). Γ is $\Gamma', A_1 \wedge A_2$, and the premise for the inference is

$$\Gamma', A_i \Vdash C,$$

where $i = 1$ or $i = 2$. By the induction hypothesis of induction, there is a deduction $\mathcal{D}$ in TM of

$$\begin{array}{c} A_i \\ \mathcal{D} \\ C, \end{array}$$

where all undischarged assumptions are in Γ'. The desired deduction of (2) is

$$\begin{array}{c} \dfrac{A_1 \wedge A_2}{A_i}\,(\wedge\mathrm{E}) \\ \mathcal{D} \\ C. \end{array}$$

Case ($\wedge$*). Here, C is $A \wedge B$, and the premises are

$$\Gamma \Vdash A \qquad\qquad \text{and} \qquad\qquad \Gamma \Vdash B.$$

By the induction hypothesis, there are deductions in TM $\mathcal{D}_1$ and $\mathcal{D}_2$ whose undischarged assumptions are all in Γ of

$$\begin{array}{c} \mathcal{D}_1 \\ A \end{array} \qquad\qquad \text{and} \qquad\qquad \begin{array}{c} \mathcal{D}_2 \\ B. \end{array}$$

Then the desired deduction of (2) is

$$\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ A & B. \end{array}}{A \wedge B.}\,(\wedge\mathrm{I})$$

Case (*$\vee$). Here, Γ is $\Gamma', A \vee B$, and the premises are

$$\Gamma', A \Vdash C \qquad\qquad \text{and} \qquad\qquad \Gamma', B \Vdash C.$$

by the induction hypothesis, there are deductions in TM whose undischarged assumptions are in Γ' of

$$\begin{array}{c} A \\ \mathcal{D}_1 \\ C \end{array} \qquad\qquad \text{and} \qquad\qquad \begin{array}{c} B \\ \mathcal{D}_2 \\ C. \end{array}$$

Then the desired deduction of (2) is

$$\dfrac{A \vee B \quad \begin{matrix} 1 \\ [A] \\ \mathcal{D}_1 \\ C \end{matrix} \quad \begin{matrix} 2 \\ [B] \\ \mathcal{D}_2 \\ C \end{matrix}}{C.} \, (\vee\mathrm{E} - 1 - 2)$$

Case ($\vee$*). Here C is $A_1 \vee A_2$, and the premise is

$$\Gamma \Vdash A_i,$$

where $i = 1$ or $i = 2$ By the hypothesis of induction, there is a deduction in TM whose undischarged assumptions are in Γ of

$$\begin{matrix} \mathcal{D} \\ A_i. \end{matrix}$$

The desired deduction of (2) is

$$\dfrac{\begin{matrix} \mathcal{D} \\ A_i \end{matrix}}{A_1 \vee A_2.} \, (\vee\mathrm{I})$$

Case (*$\supset$). Here $\Gamma$is $\Gamma', A \supset B$, and the premises are

$$\Gamma' \Vdash A \qquad \text{and} \qquad \Gamma', B \Vdash C.$$

By the hypothesis of induction, there are deductions in TM whose undischarged assumptions (except as indicated) are in Γ' of

$$\begin{matrix} \mathcal{D}_1 \\ A \end{matrix} \qquad \text{and} \qquad \begin{matrix} B \\ \mathcal{D}_2 \\ C. \end{matrix}$$

The desired deduction of (2) is

$$\begin{matrix} \dfrac{A \supset B \quad \begin{matrix} \mathcal{D}_1 \\ A \end{matrix}}{B} \, (\supset \mathrm{E}) \\ \mathcal{D}_2 \\ C. \end{matrix}$$

Case ($\supset$*). Here C is $A \supset B$, and the premise is

$$\Gamma, A \Vdash B.$$

By the induction hypothesis, there is a deduction in TM whose undischarged assumptions (except for the one indicated) are in $\Gamma$of

$$\begin{array}{c} A \\ \mathcal{D} \\ B. \end{array}$$

Then the desired deduction of (2) is

$$\dfrac{\begin{array}{c} 1 \\ [A] \\ \mathcal{D} \\ B \end{array}}{A \supset B.} \ (\supset \mathrm{I} - 1)$$

■

Theorem 10 *If* Γ *is any sequence of formulas, and if there is a normal deduction* $\mathcal{D}$ *in TM of (2), then there is a cut-free proof of (1).*

Proof By induction on the normal deduction $\mathcal{D}$.

Basis: $\mathcal{D}$ is the one-step deduction C. Then (1) is the axiom $C \Vdash C$.

Induction step: We have two main cases:

Case 1. The last rule of the inference is an I-rule. We have cases by the rule.

Subcase ($\wedge$I). Then C is $A \wedge B$ and $\mathcal{D}$ is

$$\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ A & B \end{array}}{A \wedge B,} \ (\wedge \mathrm{E})$$

and the undischarged assumptions of $\mathcal{D}_1$ and $\mathcal{D}_2$ are all in Γ. By the induction hypothesis, $\mathcal{D}_1$, and $\mathcal{D}_2$, there are cut free proofs of

$$\Gamma \Vdash A \qquad \text{and} \qquad \Gamma \Vdash B.$$

The cut-free proof of (1) is

$$\dfrac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \wedge B.} \ (\wedge^*)$$

Subcase ($\vee$I). Here C is $A_1 \vee A_2$ and $\mathcal{D}$ is

$$\cfrac{\begin{array}{c}\mathcal{D}'\\ A_i\end{array}}{A_1 \vee A_2,}\ (\vee\mathrm{I})$$

where $i = 1$ or $i = 2$ and the undischarged assumptions are all in Γ. By the induction hypothesis, there is a cut-free proof of

$$\Gamma \Vdash A_i,$$

and the cut-free proof of (1) is

$$\frac{\Gamma \Vdash A_i}{\Gamma \Vdash A_1 \vee A_2.}\ (\vee^*)$$

Subcase ($\supset$I). Here C is $A \supset B$, and $\mathcal{D}$ is

$$\cfrac{\begin{array}{c}1\\ [A]\\ \mathcal{D}_1\\ B\end{array}}{A \supset B,}\ (\supset \mathrm{I} - 1)$$

where all the undischarged assumptions are in Γ. By the induction hypothesis, there is a cut-free proof of

$$\Gamma, A \Vdash B,$$

and the desired cut-free proof of (1) is

$$\frac{\Gamma, A \Vdash B}{\Gamma \Vdash A \supset B.}\ (\supset *)$$

Case 2. The last rule of $\mathcal{D}$ is an E-rule. Then the main (left) branch of $\mathcal{D}$ consists entirely of inferencess by E-rules, and the top formula of that branch, which is the major premise of an E-rule, is not discharged. The cases are by the E-rule of which that formula is the major premise.

Subcase ($\wedge$E). $\mathcal{D}$ is

$$\begin{array}{c}\dfrac{A_1 \wedge A_2}{A_i}\ (\wedge\mathrm{E})\\ \mathcal{D}_1\\ C,\end{array}$$

where $i = 1$ or $i = 2$. By $\mathcal{D}_1$ and the induction hypothesis, there is a cut-free proof of

$$\Gamma', A_i \Vdash C,$$

where Γ' consists of all the assumptions of Γ except possibly $A_1 \wedge A_2$ (depending on whether this formula occurs elsewhere as an undischarged assumption). By (*$\wedge$), we get

$$\Gamma', A_1 \wedge A_2 \Vdash C.$$

If this is not (1), then Γ' is the same as Γ, and we get (1) by an inference by (*W).

Subcase ($\vee$E). Because of reduction step $\vee$E, we may assume without loss of generality that this is also the last inference in the main (left) branch of the deduction, so $\mathcal{D}$ is

$$\dfrac{A \vee B \quad \begin{matrix} 1 \\ [A] \\ \mathcal{D}_1 \\ C \end{matrix} \quad \begin{matrix} 2 \\ [B] \\ \mathcal{D}_2 \\ C \end{matrix}}{C.} \ (\vee\mathrm{E} - 1 - 2)$$

By the induction hypothesis, there are cut-free proofs of

$$\Gamma', A \Vdash C \qquad \text{and} \qquad \Gamma', B \Vdash C,$$

where Γ' consists of all of Γ except possibly $A \vee B$ (depending on whether $A \vee B$ occurs elsewhere in $\mathcal{D}$ as an undischarged assumption). By ($\vee$*), we get

$$\Gamma', A \vee B \Vdash C.$$

If this is not (1), then Γ' is the same as Γ, and we get (1) by an inference by (*W).

Subcase ($\supset$E). Then $\mathcal{D}$ is

$$\begin{matrix} \dfrac{A \supset B \quad \begin{matrix} \mathcal{D}_1 \\ A \end{matrix}}{B} \ (\supset \mathrm{E}) \\ \mathcal{D}_2 \\ C. \end{matrix}$$

by $\mathcal{D}_1$, $\mathcal{D}_2$ and the induction hypothesis, there are cut-free proofs of

$$\Gamma' \Vdash A \qquad\qquad \text{and} \qquad\qquad \Gamma', B \Vdash C,$$

where Γ' consists of all of Γ except possibly $A \supset B$ (depending on whether $A \supset B$ occurs elsewhere in $\mathcal{D}$ as an undischarged assumption). By (*⊃), we get

$$\Gamma', A \supset B \Vdash C.$$

If this is not (1), then Γ' is the same as Γ, and we get (1) by an inference by (*W). ■

Theorem 11 (Cut elimination) *Any proof in* LM *can be converted into a cut-free proof with the same conclusion.*

Proof Suppose (1) is provable in LM. Then by Theorem 9 there is a deduction in TM of (2). By Theorem 4, there is a normal deduction in TM of (2). Hence, by Theorem 10, there is a cut-free proof of (1). ■

This means that a proof search in TM can be carried out in LM using all the rules *except* (Cut). Let us see some examples.

Example 7 Consider the distribution of $\wedge$ over $\vee$:

$$\Vdash (A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C)).$$

If there is a proof of this, the last inference must be by (⊃*), and the premise must be

$$A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C).$$

The last inference in a proof of this might be either (*∧) or (∨*). Let us try the first. If we recall that (*∧) corresponds to the natural deduction rule (∧E), we may note that it involves eliminating a conjunction used as an undischarged assumption, and there is no reason to suppose that there may not be more than one such occurrence of the same conjunction as an undischarged assumption, we see that we want to look at

$$A \wedge (B \vee C), A \Vdash (A \wedge B) \vee (A \wedge C) \quad \text{or} \quad A \wedge (B \vee C), B \vee C \Vdash (A \wedge B) \vee (A \wedge C).$$

(This reasoning also applies whenever we proceed backward through an inference by (*$\supset$) or (*$\vee$).) Let us try the second assuming that $B \vee C$ has been introduced by (*$\vee$); the premises are

$$A\wedge(B\vee C), B\vee C, B \Vdash (A\wedge B)\vee(A\wedge C) \quad \text{and} \quad A\wedge(B\vee C), B\vee C, C \Vdash (A\wedge B)\vee(A\wedge C).$$

We now have two branches connected by 'and', so we will have to follow both. Let us, in each case, try going backward through ($\vee$*), but making a different choice in each case. Then the premises are, respectively,

$$A \wedge (B \vee C), B \vee C, B \Vdash A \wedge B \quad \text{and} \quad A \wedge (B \vee C), B \vee C, C \Vdash A \wedge C.$$

Now let us assume that each of these comes from (*$\wedge$), so that the premise are

$$A \wedge (B \vee C), B \vee C, A, B \Vdash A \wedge B \quad \text{and} \quad A \wedge (B \vee C), B \vee C, A, C \Vdash A \wedge C.$$

Finally, let us assume that each of these comes by ($\wedge$*), so that there are two premises for each. For the first the premises are

$$A \wedge (B \vee C), B \vee C, A, B \Vdash A \quad \text{and} \quad A \wedge (B \vee C), B \vee C, A, B \Vdash B,$$

whereas for the second they are

$$A \wedge (B \vee C), B \vee C, A, C \Vdash A \quad \text{and} \quad A \wedge (B \vee C), B \vee C, A, C \Vdash C.$$

Each of these last four sequents can be obtained from an axiom by repeated inferences by (*K), so we have found our proof. Eliminating unnecessary formulas, it is the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A}{A, B \Vdash A}(*K) \quad \dfrac{B \Vdash B}{A, B \Vdash B}(*K)}{A, B \Vdash A \wedge B}(\wedge^*)}{A, B \Vdash (A \wedge B) \vee (A \wedge C)}(\vee^*) \quad \dfrac{\dfrac{\dfrac{A \Vdash A}{A, C \Vdash A}(*K) \quad \dfrac{C \Vdash C}{A, C \Vdash C}(*K)}{A, C \Vdash A \wedge C}(\wedge^*)}{A, C \Vdash (A \wedge B) \vee (A \wedge C)}(\vee^*)}{\dfrac{\dfrac{\dfrac{A, B \vee C \Vdash (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C), B \vee C \Vdash (A \wedge B) \vee (A \wedge C)}(^*\wedge)}{A \wedge (B \vee C), A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C)}(^*\wedge)}{\dfrac{A \wedge (B \vee C) \Vdash (A \wedge B) \vee (A \wedge C)}{\Vdash (A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C)).}(\supset {}^*)}(^*\mathrm{W})}(^*\vee)$$

If we convert this into a natural deduction proof, we get the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{[A \wedge (B \vee C)]}}{B \vee C}\,(\wedge\mathrm{E}) \qquad \dfrac{\dfrac{\dfrac{\overset{3}{[A \wedge (B \vee C)]}}{A}\,(\wedge\mathrm{E}) \quad \overset{1}{[B]}}{A \wedge B}\,(\wedge\mathrm{I})}{(A \wedge B) \vee (A \wedge C)}\,(\vee\mathrm{I}) \qquad \dfrac{\dfrac{\dfrac{\overset{3}{[A \wedge (B \vee C)]}}{A}\,(\wedge\mathrm{E}) \quad \overset{2}{[C]}}{A \wedge C}\,(\wedge\mathrm{I})}{(A \wedge B) \vee (A \wedge C)}\,(\vee\mathrm{I})}{(A \wedge B) \vee (A \wedge C)}\,(\vee\mathrm{E}-1-2)}{(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C)).}\,(\supset\mathrm{I}-3)}{}$$

Example 8 Consider the sequent

$$\Vdash \neg A \supset (A \supset B).$$

This must come by ($\supset$*) from

(3) $$\neg A \Vdash A \supset B,$$

which probably comes by the same rule from

$$\neg A, A \Vdash B.$$

This last can only come by (*$\neg$) from

$$\neg A, A \Vdash A \qquad \text{and} \qquad A, \perp \Vdash B.$$

The first of these can come from an axiom by (*K), but not the second, so no proof can be obtained this way. The only other alternative is to try to get (3) by (*$\neg$) from

$$\neg A \Vdash A \qquad \text{and} \qquad \perp \Vdash A \supset B,$$

but the first of these is impossible, since we would have to get it by (*$\neg$) whose first premise is the same formula. So there is no proof in TM of this.

Remark 4 A system LJ corresponding to TJ can be formed by adding to LM the rule ($\perp$*):

$$\frac{\Gamma \Vdash \perp}{\Gamma \Vdash A.}$$

To prove Theorem 9 for LJ and TJ, we need to add a case for this rule; apply rule ($\perp_{\mathrm{J}}$) to the induction hypothesis. To prove Theorem 10, add a Case 3 for when the last inference of $\mathcal{D}$ is by ($\perp_{\mathrm{J}}$); in this case apply rurle ($\perp$*) to the induction hypothesis. It follows that Theorem 11 applies to LJ as well as LM. In LJ, Example 8 would work, since $A, \perp \Vdash B$ could come by ($\perp$*) from $A, \perp \Vdash \perp$, which does follow from an axiom by (*K). Hence, the formula of this example can be proved in LJ but not in LM.

6 System LK for proof searches in TK

The L-rule which corresponds to ($\perp_{\mathrm{C}}$) is

$$\frac{\Gamma, \neg A \Vdash \perp}{\Gamma \Vdash A.}$$

A rule like this would appear to interfere with the backward proof search that works so well in LM and LJ. For this reason, we formulate LK a different way: we allow sequences of more than one formula on the right hand side of the symbol '$\Vdash$'. At first, this must seem like a strange thing to do, but it is a trick that works.

Definition 7 The *system* LK is uses sequents of the form

$$\Gamma \Vdash \Delta,$$

where both Γ and Δ are sequences of formulas. The axioms are the same as those for LM. The rules are as follows (where Θ is also a sequence of formulas and $i = 1$ or $i = 2$):

(*C) $$\frac{\Gamma, A, B, \Theta \Vdash \Delta}{\Gamma, B, A, \Theta \Vdash \Delta}$$ (C*) $$\frac{\Gamma \Vdash \Delta, A, B, \Theta}{\Gamma \Vdash \Delta, B, A, \Theta}$$

(*K) $$\frac{\Gamma \Vdash \Delta}{\Gamma, A \Vdash \Delta}$$ (K*) $$\frac{\Gamma \Vdash \Delta}{\Gamma \Vdash A, \Delta}$$

(*W) $$\frac{\Gamma, A, A \Vdash \Delta}{\Gamma, A \Vdash \Delta}$$ (W*) $$\frac{\Gamma \Vdash A, A, \Delta}{\Gamma \Vdash A, \Delta}$$

(*$\wedge$) $$\frac{\Gamma, A_i \Vdash \Delta}{\Gamma, A_1 \wedge A_2 \Vdash \Delta}$$ ($\wedge$*) $$\frac{\Gamma \Vdash A_1, \Delta \quad \Gamma, A_2, \Delta}{\Gamma \Vdash A_1 \wedge A_2, \Delta}$$

$$(*\vee)\ \frac{\Gamma, A_1 \Vdash \Delta \quad \Gamma, A_2 \Vdash \Delta}{\Gamma, A_1 \vee A_2 \Vdash \Delta} \qquad (\vee *)\ \frac{\Gamma \Vdash A_i, \Delta}{\Gamma \Vdash A_1 \vee A_2, \Delta}$$

$$(*\supset)\ \frac{\Gamma \Vdash A, \Delta \quad \Gamma, B \Vdash \Delta}{\Gamma, A \supset B \Vdash \Delta} \qquad (\supset *)\ \frac{\Gamma, A \Vdash B, \Delta}{\Gamma \Vdash A \supset B, \Delta}$$

$$(\bot *)\ \frac{\Gamma \Vdash \bot, \Delta}{\Gamma \Vdash C, \Delta} \qquad (\text{Cut})\ \frac{\Gamma \Vdash A, \Delta \quad \Gamma, A \Vdash \Delta}{\Gamma \Vdash \Delta.}$$

Note the derived rules for negation:

$$(*\neg)\ \frac{\Gamma \Vdash A, \Delta \quad \Gamma, \bot \Vdash \Delta}{\Gamma, \neg A \Vdash \Delta} \qquad (\neg *)\ \frac{\Gamma, A \Vdash \bot, \Delta}{\Gamma \Vdash \neg A, \Delta}$$

Note also that we will largely ignore rule (C*) as we have been largely ignoring (*C).

Remark 5 There are presentations of LK without the rule $(\bot *)$. These are usually systems in which $\bot$ is not used at all, and it is possible to have a void sequence on the right side. Thus, the sequent that we would represent as $\Gamma \Vdash \bot$ would, in such a system, be $\Gamma \Vdash$. In such a system, rule $(\bot *)$ is a special case of (K*). This rule may seem to violate the spririt of an L-system that no formula ever disappear, but it does not really interfere with a proof search going backward, since going backward one can always try replacing a formula on the right by $\bot$.

It is worth saying a word about the interpretation of these sequents. A sequent of LM or LJ,

$$A_1, A_2, \ldots, A_n \Vdash B,$$

is interpreted as being equivalent to the formula

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset B.$$

A sequent of LK,

$$A_1, A_2, \ldots, A_n \Vdash B_1, B_2, \ldots B_m,$$

is interpreted as being equivalent to the formula

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset B_1 \vee B_2 \vee \ldots \vee B_m.$$

Example 9 Proof in LM of $\Vdash \neg A \vee A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A}{A \Vdash \bot, A}\,(\mathrm{K}^*)}{\Vdash \neg A, A}\,(\neg^*)}{\Vdash \neg A \vee A, A}\,(\vee^*)}{\Vdash \neg A \vee A, \neg A \vee A}\,(\vee^*)}{\Vdash \neg A \vee A.}\,(\mathrm{W}^*)$$

Note that we need the extra formula on the right to make this proof work.

For the following theorem, we need the following definition: If Γ is a sequence of formulas

$$A_1, A_2, \ldots, A_n,$$

then $\Gamma^\vee$ is the formula

$$A_1 \vee A_2 \vee \ldots \vee A_n.$$

If $n = 1$, then $\Gamma^\vee$ is just Γ.

Theorem 12 *If*

(4) $$\Gamma \Vdash \Delta$$

is provable in LK*, then*

(5) $$\Gamma \ \vdash \ _{\mathrm{TK}}\Delta^\vee.$$

In particular, if

$$\Gamma \Vdash C$$

is provable in LK*, then*

$$\Gamma \ \vdash \ _{\mathrm{TK}}C.$$

Proof The second sentence of the theorem follows from the first as a special case. The first sentence is proved by induction on the proof of (4).

Basis: (4) is an axiom $C \Vdash C$. Then (5) is the one-step deduction C.

Induction step: Cases by the last rule in the proof of (4). If the rule is any rule on the left except (*⊃), the proof is similar to the corresponding case in the proof of Theorem 9.

Case (C*). Here Δ is Δ_1, B, A, Δ_2, $\Delta^\vee$ is $\Delta_1^\vee \vee B \vee A \vee \Delta_2^\vee$, and the premise is

$$\Gamma \Vdash \Delta_1, A, B, \Delta_2.$$

By the induction hypothesis, there is a deduction in TK whose undischarged assumptions are all in Γ of

$$\begin{array}{c}\mathcal{D}\\ \Delta_1^{\vee} \vee A \vee B \vee \Delta_2^{\vee}.\end{array}$$

Reading $\Delta_1^{\vee} \vee A \vee B \vee \Delta_2^{\vee}$ as $((\Delta_1^{\vee} \vee A) \vee B) \vee \Delta_2^{\vee}$, we can prove (5) as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta_1^{\vee} \vee A \vee B \vee \Delta_2^{\vee}\end{array} \qquad \begin{array}{c}1\\ [\Delta_1^{\vee} \vee A \vee B]\\ \mathcal{D}_1\\ \Delta^{\vee}\end{array} \qquad \dfrac{\begin{array}{c}2\\ [\Delta_2^{\vee}]\end{array}}{\Delta^{\vee}}\,(\vee\mathrm{I})}{\Delta^{\vee},}\,(\vee\mathrm{E} - 1 - 2)$$

where $\mathcal{D}_1$ is

$$\dfrac{\Delta_1^{\vee} \vee A \vee B \qquad \begin{array}{c}3\\ [\Delta_1^{\vee} \vee A]\\ \mathcal{D}_2\\ \Delta^{\vee}\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}4\\ [B]\end{array}}{\Delta_1^{\vee} \vee B}\,(\vee\mathrm{I})}{\Delta_1^{\vee} \vee B \vee A}\,(\vee\mathrm{I})}{\Delta^{\vee}}\,(\vee\mathrm{I})}{\Delta^{\vee},}\,(\vee\mathrm{E} - 3 - 4)$$

and where $\mathcal{D}_2$ is

$$\dfrac{\Delta_1^{\vee} \vee A \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}5\\ [\Delta_1^{\vee}]\end{array}}{\Delta_1^{\vee} \vee B}\,(\vee\mathrm{I})}{\Delta_1^{\vee} \vee B \vee A}\,(\vee\mathrm{I})}{\Delta^{\vee}}\,(\vee\mathrm{I}) \qquad \dfrac{\dfrac{\begin{array}{c}6\\ [A]\end{array}}{\Delta_1^{\vee} \vee B \vee A}\,(\vee\mathrm{I})}{\Delta^{\vee}}\,(\vee\mathrm{I})}{\Delta^{\vee}.}\,(\vee\mathrm{E} - 5 - 6)$$

Case (K*). Here Δ is A, Δ_1 and the premise is

$$\Gamma \Vdash \Delta_1.$$

By the hypothesis of induction, there is a deduction in TK all of whose undischarged assumptions are in Γ of

$$\begin{array}{c}\mathcal{D}\\ \Delta_1^{\vee}\end{array}$$

Since $\Delta^\vee$ is $A \vee \Delta_1^\vee$, we get (5) as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta_1^\vee\end{array}}{\Delta^\vee.}\,(\vee\mathrm{I})$$

Case (W*). Here Δ is A, Δ_1, $\Delta^\vee$ is $A \vee \Delta_1^\vee$, and the premise is

$$\Gamma \Vdash A, A, \Delta_1.$$

By the induction hypothesis, there is a deduction in TK all of whose undischarged assumptions are in Γ of

$$\begin{array}{c}\mathcal{D}\\ A \vee A \vee \Delta_1^\vee.\end{array}$$

The proof of (5) is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A \vee A \vee \Delta_1^\vee\end{array} \qquad \dfrac{\overset{2}{[A \vee A]} \qquad \dfrac{\overset{3}{[A]}}{\Delta^\vee}\,(\vee\mathrm{I}) \qquad \dfrac{\overset{4}{[A]}}{\Delta^\vee}\,(\vee\mathrm{I})}{\Delta^\vee}\,(\vee\mathrm{E}-3-4) \qquad \dfrac{\overset{2}{[\Delta_1^\vee]}}{\Delta^\vee}\,(\vee\mathrm{I})}{\Delta^\vee.}\,(\vee\mathrm{E}-1-2)$$

Case ($\wedge$*). Here, Δ is $A_1 \wedge A_2, \Delta_1$, so $\Delta^\vee$ is $(A_1 \wedge A_2) \vee \Delta_1^\vee$, and the premises are

$$\Gamma \Vdash A_1, \Delta \qquad\qquad \text{and} \qquad\qquad \Gamma \Vdash A_2, \Delta.$$

By the induction hypothesis, there are deductions in TK all of whose undischarged assumptions are in Γ of

$$\begin{array}{c}\mathcal{D}_1\\ A_1 \vee \Delta_1^\vee\end{array} \qquad\qquad \text{and} \qquad\qquad \begin{array}{c}\mathcal{D}_2\\ A_2 \vee \Delta_1^\vee.\end{array}$$

The deduction of (5) is then as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A_1 \vee \Delta_1^\vee\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ A_2 \vee \Delta_1^\vee\end{array} \qquad \dfrac{\dfrac{\overset{1}{[A_1]} \quad \overset{3}{[A_2]}}{A_1 \wedge A_2}\,(\wedge\mathrm{I})}{\Delta^\vee}\,(\vee\mathrm{I}) \qquad \dfrac{\overset{4}{[\Delta_1^\vee]}}{\Delta^\vee}\,(\vee\mathrm{I})}{\Delta^\vee}\,(\vee\mathrm{E}-3-4) \qquad \dfrac{\overset{2}{[\Delta_1^\vee]}}{\Delta^\vee}\,(\vee\mathrm{I})}{\Delta^\vee.}\,(\vee\mathrm{E}-1-2)$$

Case ($\vee$*). Δ is $A_1 \vee A_2, \Delta_1$, so $\Delta^\vee$ is $(A_1 \vee A_2) \vee \Delta_1^\vee$, and the premise is

$$\Gamma \Vdash A_i, \Delta_1,$$

where $i = 1$ or $i = 2$. By the induction hypothesis, there is a deduction in TK all of whose undischarged assumptions are in Γ of

$$\begin{array}{c} \mathcal{D} \\ A_i \vee \Delta_1^\vee. \end{array}$$

The deduction of (5) is as follows:

$$\dfrac{\begin{array}{c} \mathcal{D} \\ A_i \vee \Delta_1^\vee. \end{array} \qquad \dfrac{\dfrac{\overset{1}{[A_i]}}{A_1 \vee A_2}\,(\vee\mathrm{I})}{\Delta^\vee}\,(\vee\mathrm{I}) \qquad \dfrac{\overset{2}{[\Delta_1^\vee]}}{\Delta^\vee}\,(\vee\mathrm{I})}{\Delta^\vee.}\,(\vee\mathrm{E}-1-2)$$

Case (*$\supset$). Here Γ is $\Gamma_1, A \supset B$, and the premises are

$$\Gamma_1 \Vdash A, \Delta \qquad \text{and} \qquad \Gamma_1, B \Vdash \Delta.$$

By the induction hypothesis, there are deductions in TK all of whose undischarged assumptions (except for B in the second one) are in Γ_1 of

$$\begin{array}{c} \mathcal{D}_1 \\ A \vee \Delta^\vee \end{array} \qquad \text{and} \qquad \begin{array}{c} B \\ \mathcal{D}_2 \\ \Delta^\vee. \end{array}$$

Then the deduction of (5) is

$$\dfrac{\begin{array}{c} \mathcal{D}_1 \\ A \vee \Delta^\vee \end{array} \qquad \begin{array}{c} \dfrac{A \supset B \quad \overset{1}{[A]}}{B}\,(\supset\mathrm{E}) \\ \mathcal{D}_2 \\ \Delta^\vee \end{array} \qquad \overset{2}{[\Delta^\vee]}}{\Delta^\vee.}\,(\vee\mathrm{E}-1-2)$$

Case ($\supset$*). Δ is $A \supset B, \Delta_1$, so $\Delta^\vee$ is $(A \supset B) \vee \Delta_1^\vee$, and the premise is

$$\Gamma, A \Vdash B, \Delta_1.$$

By the induction hypothesis, there is a deduction in TK all of whose undischarged assumptions (except the indicated occurrence of A) are in Γ of

$$\begin{array}{c} A \\ \mathcal{D} \\ B \vee \Delta_1^\vee. \end{array}$$

Then the deduction of (5) is

$$\begin{prooftree}
\AxiomC{$\overset{5}{[\neg\Delta^\vee]}$}
\AxiomC{$\overset{2}{[\neg A]}$}
\AxiomC{$\overset{1}{[A]}$}
\RightLabel{$(\neg\mathrm{E})$}
\BinaryInfC{$\perp$}
\RightLabel{$(\perp_{\mathrm{C}} - \mathrm{v})$}
\UnaryInfC{B}
\RightLabel{$(\supset \mathrm{I} - 1)$}
\UnaryInfC{$A \supset B$}
\RightLabel{$(\vee\mathrm{I})$}
\UnaryInfC{$\Delta^\vee$}
\RightLabel{$(\neg\mathrm{E})$}
\BinaryInfC{$\perp$}
\RightLabel{$(\perp_{\mathrm{C}} - 2)$}
\UnaryInfC{A}
\noLine
\UnaryInfC{$\mathcal{D}$}
\noLine
\UnaryInfC{$B \vee \Delta_1^\vee$}
\AxiomC{$\overset{3}{[B]}$}
\RightLabel{$(\supset \mathrm{I} - \mathrm{v})$}
\UnaryInfC{$A \supset B$}
\RightLabel{$(\vee\mathrm{I})$}
\UnaryInfC{$\Delta^\vee$}
\AxiomC{$\overset{4}{[\Delta_1^\vee]}$}
\RightLabel{$(\vee\mathrm{I})$}
\UnaryInfC{$\Delta^\vee$}
\RightLabel{$(\vee\mathrm{E} - 3 - 4)$}
\TrinaryInfC{$\Delta^\vee$}
\RightLabel{$(\perp_{\mathrm{C}} - 5)$}
\UnaryInfC{$\Delta^\vee.$}
\end{prooftree}$$

(Note that we have here two inferences by $(\perp_{\mathrm{C}})$ which discharge assumptions nonvacuously, and one of them is not the last inference of the deduction. This is because we cannot push the first of them pas the rules of $\mathcal{D}$ without changing it, and this would complicate the presentation.)

Case $(\perp^*)$. Δ is C, Δ_1, so $\Delta^\vee$ is $C \vee \Delta_1^\vee$, and the premise is

$$\Gamma \Vdash \perp, \Delta_1.$$

By the induction hypothesis, there is a deduction in TK all of whose undischarged assumptions are in Γ of

$$\begin{array}{c} \mathcal{D} \\ \perp \vee \Delta_1^\vee. \end{array}$$

Then the deduction of (5) is

$$\begin{prooftree}
\AxiomC{$\mathcal{D}$}
\noLine
\UnaryInfC{$\perp \vee \Delta_1^\vee$}
\AxiomC{$\overset{1}{[\perp]}$}
\RightLabel{$(\perp_{\mathrm{C}} - \mathrm{v})$}
\UnaryInfC{$\Delta^\vee$}
\AxiomC{$\overset{2}{[\Delta_1^\vee]}$}
\RightLabel{$(\vee\mathrm{I})$}
\UnaryInfC{$\Delta^\vee$}
\RightLabel{$(\vee\mathrm{E} - 1 - 2)$}
\TrinaryInfC{$\Delta^\vee.$}
\end{prooftree}$$

Case (Cut). The premises are

$$\Gamma \Vdash A, \Delta \qquad\qquad \text{and} \qquad\qquad \Gamma, A \Vdash \Delta.$$

By the induction hypthesis, there are deductions in TK whose undischarged assumptions (except for the indicated occurrence of A) are in Γ of

$$\begin{array}{c} \mathcal{D}_1 \\ A \vee \Delta^{\vee} \end{array} \qquad\qquad \text{and} \qquad\qquad \begin{array}{c} A \\ \mathcal{D}_2 \\ \Delta^{\vee}. \end{array}$$

The deduction of (5) is

$$\dfrac{\begin{array}{c} \mathcal{D}_1 \\ A \vee \Delta^{\vee} \end{array} \quad \begin{array}{c} 1 \\ [A] \\ \mathcal{D}_2 \\ \Delta^{\vee} \end{array} \quad \begin{array}{c} 2 \\ [\Delta^{\vee}] \end{array}}{\Delta^{\vee}.} \ (\vee\mathrm{E} - 1 - 2)$$

■

Theorem 13 *If there is a normal deduction $\mathcal{D}$ of*

$$\Gamma \ \vdash \ _{\mathrm{TK}} C$$

then there is a cut-free proof in LK *of*

$$\Gamma \Vdash C.$$

Proof Recall that we are normalizing TK by the second method of Remark 2. This means that we are assuming that a normal TK deduction has only one inference by rule ($\perp_{\mathrm{C}}$) that discharges assumptions nonvaciously, and that one is at the end of the deduction. This means that to the proof of Theorem 10, we need only add cases for rule ($\perp_{\mathrm{C}}$), one vacuous and one nonvacuous, and we can assume that the inference in the nonvacuous case is the last of the deduction. The vacuous case can be handled as indicated in Remark 4.

This means that for the nonvacuous case, $\mathcal{D}$ has the form

$$\dfrac{\begin{array}{c} 1 \\ [\neg C] \\ \mathcal{D}_1 \\ \perp \end{array}}{C.} \ (\perp_{\mathrm{C}} - 1)$$

Hence, by the hypothesis of induction, there is a cut-free proof in LK of

$$\Gamma, \neg C \Vdash \bot. \tag{6}$$

To complete the proof, it is sufficient to prove that if there is a cut-free proof in LK of (6), then there is a cut-free proof in LK of

$$\Gamma \Vdash C. \tag{7}$$

To carry out the proof, we need some terminology: in any rule except (Cut), the new formula introduced into the conclusion of the rule is the *principal formula*, the formula(s) from the premise(s) which appear in the conclusion only as part of the principal formula are the *side formulas*, and the formulas in Γ and Δ which are the same in premises and the conclusion are called *parameters*. Note that any inference remains valid if the parameters are changed. For example, in the rule ($\supset$I), the principal formula is $A \supset B$, the side formulas are A and B, and the formulas in Γ and Δ are parameters. Also, in rule (*K), the main formula is A, there are no side formulas, and the parameters are the formulas in Γ and Δ. Given a formula in the conclusion of an inference, its *immediate ancestors* are defined as follows: (1) if the formula is a principal formula, than any side formula is an immediate ancestor, and (2) if the formula is a parameter, then a corresponding parameter in each premise is an immediate ancestor. An *immediate parametric ancestor* is defined using only (2) from the above definition. An *ancestor* (respectively *parametric ancestor*) is the transitive closure of the relation of immediate ancestor (respectively immediate parametric ancestor). A quasi-parametric ancestor is defined similarly except that principal formulas and side formulas of (*W*) and (*C*) are allowed. Note that a parametric or quasi-parametric ancestor of a formula is identical to the formula.

Now let $E_1, E_2, \ldots, E_n$ be a proof of (6) in which the sequents E_i are so ordered that each one except E_n is used exactly once as a premise for an inference, and let E_i be

$$\Gamma_i, U_i \Vdash V_i, \Delta_i,$$

where U_i consists of the quasi-parametric ancestors of the indicated occurrence of $\neg C$ in (6) and V_i consists of the quasi-parametric ancestors of the indicated occurrence of $\bot$ in (6). Let E_i' be

$$\Gamma_i \Vdash C, \Delta_i.$$

Since Γ_n is Γ and Δ_i is the empty sequence of formulas, E'_n is (7), so our proof will be complete if we prove E'_i for each $i = 1, 2, \ldots, n$. This will be proved by induction on i. The case for the basis will be distributed among the cases for the induction step.

Case 1. Both U_i and V_i are void. Then E_i is $\Gamma_i \Vdash \Delta_i$, and E'_i follows from E_i by (K*).

Case 2. U_i and V_i are not both void, and E_i is an axiom. Then only one of U_i and V_i is nonvoid.

Subcase 2.1. U_i is nonvoid, so V_i is void. Then Γ_i is void and Δ_i is $\neg C$, so E_i is $\neg C \Vdash \neg C$. Then E'_i is $\Vdash \neg C, C$, which is proved as follows:

$$\dfrac{\dfrac{C \Vdash C}{C \Vdash \bot, C}\,(\mathrm{K}*)}{\Vdash \neg C, C.}\,(\neg^*)$$

Subcase 2.2. V_i is nonvoid. Then U_i and Δ_i are void, V_i consists of a single occurrence of $\bot$, and Γ also consists of $\bot$, so that E_i is $\bot \Vdash \bot$. Then E'_i is $\bot \Vdash C$, and its proof is as follows:

$$\dfrac{\bot \Vdash \bot}{\bot \Vdash C.}\,(\bot^*)$$

Case 3. U_i and V_i are not both void, and E_i is obtained from E_j and perhaps E_k by a rule for which all the formulas in U_i and V_i are parametric. Then the same inference (with different parameters) will permit E'_i to be obtained from E'_j and perhaps E'_k.

Case 4. U_i and V_i are not both void, and E_i is obtained from E_j by a structural rule whose principal constituent is in U_i or V_i. This inference only changes the number of occurrences of formulas in U_i or V_i, so E'_i is the same as E'_j or, if the number is changed from 0 to 1, E'_i follows by one of (*K*) from E'_j.

Case 5. U_i and V_i are not both void, and E_i is obtained by an inference by an operational rule for which the principal formula is in U_i or V_i. Since the principal formula of an operational rule cannot be $\bot$, it is not in V_i, but must be in U_i. Since every formula in U_i is $\neg C$, the rule must be (*$\neg$), and there are two premises, say E_j and E_k. Let U'_i be the formulas of U_i which are not the principal formula of the inference. Then E_i is $\Gamma_i, U'_i, \neg C \Vdash V_i, \Delta_i$. The left premise, E_j, is $\Gamma_i, U'_i \Vdash C, V_i, \Delta_i$, so Γ_j is Γ_i, U_j is U'_i, V_j is V_i, and

Δ_j is C, Δ_i. It follows that E'_j, which is $\Gamma_j \Vdash C, \Delta_j$, is $\Gamma_i \Vdash C, C, \Delta_i$, so that E'_i, which is $\Gamma_i \Vdash C, \Delta_i$, can be obtained from E'_j by an inference by (W*). ■

Theorem 14 (Cut elimination) *Any proof in LK can be replaced by a cut-free proof of the same conclusion.*

Proof Like the proof of Theorem 11. ■

Let us now look at some examples of proof search in LK.

Example 10 Consider

$$\Vdash ((A \supset B) \supset A) \supset A.$$

This might come by ($\supset$*) from

$$(A \supset B) \supset A \Vdash A.$$

This, in turn, must come by (*$\supset$) from

$$(A \supset B) \supset A \Vdash A \supset B, A \qquad\qquad \text{and} \qquad\qquad A \Vdash A.$$

The second of these is an axiom, while the first can come by ($\supset$*) from

$$(A \supset B) \supset A, A \Vdash B, A,$$

and this can come from an axiom by (*K*). Thus, we have the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Vdash A}{A \Vdash B, A}\,(\mathrm{K}^*)}{\Vdash A \supset B, A}\,(\supset *) \qquad A \Vdash A}{(A \supset B) \supset A \Vdash A}\,(* \supset)}{\Vdash ((A \supset B) \supset A) \supset A.}\,(\supset *)$$

Note that this proof depends on having more than one formula on the right side, and so it will not work in either LJ or LM.

Example 11 Consider

$$A \vee B \Vdash A \wedge B.$$

This might come by ($\wedge$*) from

$$A \vee B \Vdash A \qquad \text{and} \qquad A \vee B \Vdash B.$$

Each of these might come from (*$\vee$). The premises for the first are

$$A \Vdash A \qquad \text{and} \qquad B \Vdash A,$$

while the premises for the second are

$$A \Vdash B \qquad \text{and} \qquad B \Vdash B.$$

Although two of these four are axioms, the other two cannot be axioms. Hence this cannot be proved in LK. And we would not expect it to be provable in LK, since there is a truth assignment which makes the left side true and the right side false. We can assign T to A and F to B, for example. And this assignment will falsify the branch of our search leading to $A \Vdash B$.

We can now prove that every tautology is provable in LK (and hence in TK). We will do this by proving that if a sequent $\Gamma \Vdash \Delta$ is not provable in LK, then there is a truth assignment which falsifies it in the sense of assigning T to every formula in Γ and F to every formula in Δ. It will follow that if any assignment which assigns every formula in Γ the value T also assigns T to at least one formula in Δ, then $\Gamma \Vdash \Delta$ must be provable.

So suppose $\Gamma \Vdash \Delta$ is not provable in LK. Then there is a branch of our proof search which ends in a sequent in which all the "atomic" letters appear, but none appears on both sides. Assign T to every "atomic" letter on the left of that sequent and F to every "atomic" letter on the right, but always assign F to $\bot$. By the truth tables, this will assign T to every formula on the left and F to every formula on the right. This assignment falsifies that sequent. An examination of the rules shows that if an assignment falsifies a premise for any inference, it can be extended to one that falsifies the conclusion. Then this assignment will falsify $\Gamma \Vdash \Delta$. This proves

Theorem 15 (Completeness) *If there is no assignment of truth values to the formulas of a sequent which falsifies the sequent, then the sequent is provable in* LK.

Corollary 15.1 *Every tautology is provable in* LK *(and hence also in* TK*).*

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