Curry's Formalism as Structuralism*

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Abstract

In 1939, Curry proposed a philosophy of mathematics he called *formalism*. He made this proposal in two works originally written in 1939. These are the two philosophical works for which Curry is known, and they have left a false impression of his views. In this article, I propose to clarify Curry's views by referring to some of his later writings on the subject. I claim that Curry's philosophy was not what is now usually called formalism, but is really a form of structuralism.

1 Curry's early philosophy of mathematics

In [2], which is a shortened form of the original manuscript of [4], Curry proposed a philosophy of mathematics he called *formalism*. These two works, which represent Curry's views in 1939, early in his career, are often the only works by which Curry's philosophical ideas are known; see, for example, [15, Chapter 6, §5], where [4] is mistakenly identified as a mature work.

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In these early works, Curry proposed defining mathematics as the *science of formal systems*.

For some writers on the philosophy of mathematics, formalism seems to mean that all mathematics can be interpreted inside a formal system. See, for example, [9].[1]

But this was not Curry's view. In [4, p. 56], Curry made this clear:

> This definition should be taken in a very general sense. The incompleteness theorems mentioned at the close of Chapter IX[2] show that it is hopeless to find a single formal system which will include all of mathematics as ordinarily understood. Moreover the arbitrary nature of the definitions which can constitute the primitive frame[3] of a formal system shows that, in principle at least, all formal systems stand on a par. The essence of mathematics lies, therefore, not in any particular kind of formal system but in formal structure as such. The considerations of the preceding section show furthermore that we must include metapropositions as well as elementary ones. Indeed, all propositions having to do with one formal system or several or with formal systems in general are to be regarded as purely mathematical in so far as their criteria of truth depend on formal considerations alone, and not on extraneous matters.

Thus, for Curry, formalism meant that mathematics could be taken to be statements *about* formal systems, or, in other words, the metatheory of formal systems. In fact, Curry's notion of formal system was different from the usual one, and did not need to include the connectives and quantifiers of logic. Consider, for example, the following formal system for the natural numbers:[4]

Example 1 The formal system $\mathcal{N}$ is defined as follows:

- Atomic term:[5] 0

[1] This idea seems to be related to Hilbert's program of proof theory.

[2] The preceding chapter, where Curry referred to Gödel's Incompleteness Theorems.

[3] Curry used this term to refer to the basic definition of a formal system. Example 1 below gives the primitive frame for the formal system $\mathcal{N}$.

[4] This is the system called $\mathcal{N}_1$ in [14], which, in turn, is the system of [5, p.256] and of [4, Example 1, page 18].

[5] In [4] atomic terms are called *tokens*. In Curry's later works, they are called *atoms*.

- Primitive (term forming) operation: forms $t|$ from t
- Terms: $0|| \dots |$
- Primitive predicate: $=$

  Elementary statements: $t_1 = t_2$
- Axiom: $0 = 0$
- Rule: $t_1 = t_2 \Rightarrow t_1| = t_2|$

In terms of Curry's ideas, most classical mathematics can be obtained as part of the metatheory of this formal system provided that one allows sufficiently strong methods of proof in the metatheory. Curry never expected it to be possible to obtain all mathematics in this way from one formal system, but he did assume in 1939 that for any part of mathematics, a formal system could be found for which that part of mathematics would be part of the metatheory.

Note that the terms of this system are not necessarily strings of symbols. In the definition, the terms are not exhibited, but only referred to by names. The formal system is abstract in this sense. One feature of it is that every term has a unique construction from the atomic term by the term forming operator(s). Curry noted in [3] that his idea of a formal system differed from the notion used by most logicians, which is also called a *calculus*, and is based on strings of characters.[6]

In communicating information about a formal system, it is necessary to use language. Since a calculus is, by definition, part of a language, it can be easily discussed using the symbols to name themselves. But for a formal system in Curry's sense, which is not by its nature linguistic, it is necessary to relate it to a language in some way. Curry used English the way mathematicians do, by adding extra symbols, in order to discuss formal systems. He defined the *representation* of a formal system to be a kind of interpretation in which the terms are interpreted in some way but the primitive predicates are defined by the axioms and rules of the system.[7]

[6] These strings of characters are part of what is usually called the *object language*, whereas the metatheory is expressed in what is usually called the *metalanguage*.

[7] In other words, the terms are interpreted externally to the formal system, but the truth of the elementary statements depends only on the axioms and rules, i.e., on the definition of the formal system.

The symbols Curry used to discuss the formal system would thus form a representation of the system. Curry reserved the word '*interpretation*' of a formal system for the situation in which the predicates and elementary statements are also interpreted outside the system, so that there may be interpretations in which theorems of the formal system are not true.

Curry contrasted his formalism with what he called *contensivism*:[8] the idea that mathematics has a definite subject matter that exists prior to any mathematical activity. This is obvious for platonism; for intuitionism, the subject matter of mathematics was the mental constructions involved. Curry's idea was that the only subject matter of any mathematics is created by mathematical activity, for example by defining a formal system. This meant that mathematics is characterized by its *method*, and the objects to which that method is applied are usually left unspecified.

The name of formalism has a reputation of referring to the manipulation of meaningless symbols. This was never Curry's view. Beginning with [1], Curry argued that the statements of mathematics have meaning.[9] For Curry, metatheoretic statements about a formal system were meaningful statements about that system. Indeed, Curry regarded mathematics as being like language, a creation of human beings. In Karl Popper's terms, he regarded mathematics as part of the third world. I have some personal evidence that Curry took this view: when Popper presented his paper [13] in Amsterdam in 1967, Curry was chair of the session. After the session was over, Curry told me privately that in his opinion Popper had made too much of something that was obviously and almost trivially true.

With regard to the question of truth of mathematical statements, Curry differentiated between:

1. Truth within a theory (or formal system): determined by the definition of the theory.

2. The acceptability of a theory (or formal system) for some purpose.

Curry's view of what means of proof should be allowed in the metatheory was thus based on pragmatism. In much of his own work, he made the metatheory constructive because he thought this would make it acceptable

[8]Curry coined the word "contensive" to translate the German word *inhaltlich*.

[9]But by the end of his career, he regarded his original arguments as "puerile."

to more mathematicians.[10] On the other hand, he had no trouble using classical logic in the metatheory of analysis, his reason being that classical analysis is useful in physics.

2 Criticisms of Curry's early philosophy of mathematics

The main criticism of Curry's early definition of mathematics is that it implies that there was no mathematics before formal systems were first introduced in the late nineteenth century. This criticism is made in [15, p. 170], but I heard it made orally at a logic colloquium in Hannover in 1966. Curry's early definition of mathematics as the science of formal systems does not account for informal mathematics, which is most of the mathematics done in recorded history.

Another criticism is that Curry's notion of formal system is not the standard one. This fact has led to much confusion.

Additional confusion about Curry's ideas is his idiosyncratic vocabulary. Curry wanted to avoid arguments over the use of words. As a result, whenever his use of a word was criticized, he would choose another word, often one he made up himself. Thus, for example, when Kleene in [10] criticized Curry's use of the prefix "meta-" on the grounds that the prefix should only apply when the formal system was defined as strings of characters on an alphabet and not to Curry's kind of formal system, Curry decided to use the prefix "epi-" instead, and he used this prefix for the rest of his career, speaking of epitheory instead of metatheory.

Another example of Curry's idiosyncratic vocabulary resulted from his desire to have one word that could be used for the formal objects of all formal systems. The word "term" will do for combinatory logic, λ-calculus, and the system $\mathcal{N}$ of Example 1 above, but for systems of predicate logic what would be called the terms under this usage would be what are usually called formulas, and there are other formal objects called terms. For this reason, Curry coined the word "ob" (the first syllable of the word "object") for the formal objects of his systems.

Another reason that there has been confusion about Curry's ideas is that

[10] In taking this view, Curry missed the fact that to most mathematicians, who know nothing about constructivism, constructive proofs may seem strange.

Curry was not a good expository writer, and he knew it. This is why he brought Robert Feys in as a co-author of [7]. In addition, Curry criticized an early draft of my expository paper [14] on the grounds that it sounded too much like him.

3 Curry's later philosophy of mathematics

In the preface to [4], Curry says explicitly that the book represents his views as of 1939 and not the views he would defend in 1951. In fact, Curry's ideas continued to evolve throughout his career. Unfortunately, his later philosophical publications are scattered in a number of different journals and books, and so they are not as well known as his writings of 1939.

The first occasion on which he published a modification of mathematics occurred in 1963 in [5, §1C, p. 14], where he says

> ... the species of formalism here adopted maintains that the essence of mathematics lies in the formal methods as such, and that it admits all sorts of formal theories as well as general and comparative discussions reegarding the relations of formal theories to one another and to other doctrines. In this sense *mathematics is the science of formal methods.* (Emphasis in the original.)

The idea expressed here may be more general than Curry intended at the time he wrote this. For when we put a pile of five oranges in one-to-one correspondence with a pile of five apples, there is a sense in which we are being formal: we are abstracting from the specific content of the piles (i.e., the objects of which they are composed) and concentrating on their common form (i.e., the cardinal number of the piles). Thus, talking about cardinal numbers involves a formal method. Furthermore, what is true about cardinal numbers originates with a kind of formal definition: two collections have the same cardinal number just when there is a one-to-one correspondence between them.[11] This definition is part of mathematics itself, and thus satisfies one of Curry's criteria for formalism, namely that mathematics has no other subject matter than other mathematics. Thus, this definition allows for the existence of mathematics before the introduction of formal systems. Whether Curry had this idea in mind in 1963 I do not know.

[11] This definition is explicit in the work of Cantor, but it seems implicit in the process of counting, which began in prehistory.

At this time, Curry also extended his definition of formal system to include calculi. He called calculi *syntactical formal systems*, whereas he referred to formal systems of his original kind as *ob systems*. The distinction was that the formal objects of syntactical formal systems were strings of characters on some alphabet, while the formal objects of ob systems each had a unique construction from the atoms by the operations. He noted that many formal systems are of both kinds; the usual systems of propositional or predicate calculus have as formal objects formulas which are strings of characters, but the standard definition of a well-formed formula is such that each well-formed formula has a unique construction.

Later in the 1960's, in [6, p. 363], Curry wrote something which also sheds light on his view of mathematics:

> The foundations of mathematics are not like those of a building, which may collapse if its foundations fail; but they are rather more like the roots of a tree, which grow as the tree grows, and in due proportion. Thus, we can conceive of mathematics as a science which grows, as other sciences do, as much by reformation of its fundamental structure as by extension of its gross size.

Note how different this approach is from that of Henle, which requires that the formal systems in which he believes a mathematician operates function very much like the foundations of a building.

In the same paper, [6, p. 365], he modified his definition of mathematics still further, writing "...if one regards mathematics as the science of formal structure as such, and not of any particular formal structure, ..." This makes Curry's approach to mathematics look like structuralism as defined by Shapiro, who says [15, Chapter 10, p. 257] that the slogan of structuralism is, "mathematics is the science of structure."

But what are the structures to which Shapiro refers, and how do they relate to Curry's formal structures? Shapiro's explanation is [15, p. 259]:

> Define a *system* to be a collection of objects with certain relations among them. A corporate hierarchy or a government is a system of people with supervisory and co-worker relationships; a chess configuration is a system of pieces under spacial and 'possible move' relationships; a language is a system of characters, words, and sentences with syntactic and semantic relations between them; and a basketball defence is a collection of people

> with spatial and 'defensive role' relations. Define a *pattern* or *structure* to be the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system. (Emphasis in the original.)

This process of passing to the abstract form of a system and ignoring features that do not affect how they relate to other objects in the system seems to make these structures formal in Curry's sense, both in the abstraction from the particular objects involved and in the fact that in defining a structure, what is true of objects in the structure depends only on the way the structure is defined (i.e., on the way the relations are specified for that specific structure), and not on any external "subject matter."

A further sign of the similarity between what Shaprio says about structuralism and what Curry says about formalism can be seen in what Shapiro [15, p. 258] says about natural numbers:

> The structuralist vigorously rejects any sort of ontological independence among the natural numbers. The essence of a natural number is its *relations* to other natural numbers. The subject-matter of arithmetic is a single abstract structure, the pattern common to any infinite collection of objects that has a successor relation, a unique initial object, and satisfies the induction principle. The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (Emphasis in the original.)

Note how similar this is to what Curry says about natural numbers [5, p. 12]:

> A formalist would not speak of "the natural numbers" but of a set or system of natural numbers. Any system of objects, no matter what, which is generated from a certain initial object by a certain unary operation in such a way that each newly generated object is distinct from all those previously formed and that the process can be continued indefinitely, will do as a set of natural numbers.

> He may, and usually does, objectify this process by representing the numbers in terms of symbols; he chooses some symbol, let us say a vertical stroke '|', for the initial object, and regards the operation as the affixing of another '|' to the right of the given expression. But he realizes there are other interpretations; in particular, if one accepts the platonist or intuitionist metaphysics, their systems will do perfectly well.

Note how similar the two approaches are.

From what Curry told me during his lifetime, his use of the word "formalism" for his philosophy of mathematics is partly a consequence of the fact that he got his doctorate under Hilbert. But Curry's philosophy of mathematics is really not the same as any kind of formalism associated with Hilbert or his program. I think that Curry's use of the word "formalism" is unfortunate, and that Curry should really be classed among the structuralists.

Appendix

A Criticisms of Henle's formalism

A criticism of Henle's formalism was made by Miriam Lipschütz-Yevick in [11, 12]. Her criticism is that there is "...a fundamental duality in the modes of designating and recognizing objects of a formal system which called for an 'understanding' *ab initio* as to the mode in which the objects are to be viewed. I hence maintained that formal systems are no more formal and context-free than are the systems that are to be embedded in it (*sic*)."[12] The fundamental dualism is the difference between exhibiting and describing an object. Lipschütz-Yevick maintains that both of these modes of indication are needed simultaneously in dealing with formal systems (as well as non-formal mathematical theories), but that the results of these modes are not isomorphic, and that this difference undermines the claims of formalists.

This criticism might apply to formalists such as Henle, but it is irrelevant to Curry's formalism. Curry's notion of formalism does not require that there be a context-free description of formal systems. Curry is, after all, willing to

[12][12, p. 109], referring to [11]. [11] is an answer to Henle [9]. Lipschütz-Yevick errs in [12, p. 109] in saying that Henle argued in favor of Curry's formalism. As I pointed out above, Henle's formalism is very different from Curry's.

allow any reasonable means of proof in his metatheory, and which means of proof he will allow in the metatheory of a given formal system will depend on the formal system and the purpose for which it is being used.

Furthermore, Curry, near the end of his career, explicitly recognized that his description of a formal system is not context-free. Part of his definition is that the classes of atoms, primitive operations, terms, elementary statements, axioms, and applications of rules of inference be *definite*, i.e., that it be possible to determine mechanically (i.e., by an idealized computer which has no limits on memory or time) whether or not something is in one of these classes. In the late 1960s, a student attending Curry's lectures in Amsterdam pointed out that for these classes of objects to be definite in this sense, there must be a suitable universe of discourse. Even the class of words (strings of characters) on a finite alphabet is only definite in relation to words of some larger alphabet. For more details on this, see [8, §11A, p. 7]. This means that Curry explicitly recognized that his definitions and descriptions of formal systems are not and cannot be context-free. This fact has no effect on Curry's philosophy of mathematics.

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