What is a superrigid subgroup?

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Abstract

It is not difficult to see that every group homomorphism from $\mathbb{Z}^k$ to $\mathbb{R}^n$ extends to a homomorphism from $\mathbb{R}^k$ to $\mathbb{R}^n$. (Essentially, this is the fact that a linear transformation can be defined to have any desired action on a basis.) We will see other examples of discrete subgroups $\Gamma$ of connected groups $G$, such that the homomorphisms defined on $\Gamma$ can (“almost”) be extended to homomorphisms defined on all of $G$. 
What is a superrigid subgroup?

(1. Combinatorial superrigidity)

2. Group-theoretic superrigidity

(3. The analogy)

4. Superrigid subgroups
(1. Combinatorial superrigidity)

Eg. Two joined triangles

This is not rigid.

I.e., it can be deformed (a “hinge”).
Eg. Tetrahedron

This is **rigid** (cannot be deformed).
Eg. Add a small tetrahedron

This is rigid.
However, it is not superrigid: if it is taken apart, it can be reassembled incorrectly.
A tetrahedron is superrigid: the combinatorial description determines the geometric structure.

**Combinatorial superrigidity:**

Make a copy of the object,
according to the combinatorial rules.
The copy is the exact same shape as the original.

*This talk:* analogue in group theory
2. Group-theoretic superrigidity

Group homomorphism $\phi: \mathbb{Z} \to \mathbb{R}^d$

(i.e., $\phi(m + n) = \phi(m) + \phi(n)$)

$\Rightarrow \phi$ extends to a homomorphism $\hat{\phi}: \mathbb{R} \to \mathbb{R}^d$.

Namely, define $\hat{\phi}(x) = x \cdot \phi(1)$.

Check:

• $\hat{\phi}(n) = \phi(n)$

• $\hat{\phi}(x + y) = \hat{\phi}(x) + \hat{\phi}(y)$

• $\hat{\phi}$ is continuous

(only allow continuous homomorphisms)
Group homomorphism $\phi: \mathbb{Z}^k \to \mathbb{R}^d$

$\Rightarrow \phi$ extends to a homomorphism $\hat{\phi}: \mathbb{R}^k \to \mathbb{R}^d$.

**Proof.** Use standard basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$.

“A linear transformation can have any desired effect on a basis.”

Linear transformation

$\Rightarrow$ homomorphism of additive groups

**Group Representation Theory:**
study homomorphisms into *Matrix Groups*.

$\text{GL}_d(\mathbb{C}) = d \times d$ matrices over $\mathbb{C}$

with nonzero determinant

This is a group under multiplication.

$$\mathbb{R}^d \cong \begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Group homomorphism $\phi: \mathbb{Z} \to \text{GL}_d(\mathbb{R})$
(i.e., $\phi(m + n) = \phi(m) \cdot \phi(n)$)

$\not\Rightarrow$ extends to homo $\hat{\phi}: \mathbb{R} \to \text{GL}_d(\mathbb{R})$.

(Only allow continuous homos.)

*Proof by contradiction.*
Suppose $\exists$ homo $\hat{\phi}: \mathbb{R} \to \text{GL}_d(\mathbb{R})$ with $\hat{\phi}(n) = \phi(n)$ for all $n \in \mathbb{Z}$.

$\hat{\phi}(0) = I \Rightarrow \det(\hat{\phi}(0)) = 1 > 0$

$\mathbb{R}$ connected

$\Rightarrow \hat{\phi}(\mathbb{R})$ connected

$\Rightarrow \det(\hat{\phi}(\mathbb{R}))$ connected

$\Rightarrow \det(\hat{\phi}(\mathbb{R})) > 0$

$\Rightarrow \det(\phi(1)) > 0$

Maybe $\det(\phi(1)) < 0$. \hspace{1cm} \rightarrow \leftarrow

(Any $A \in \text{GL}_d(\mathbb{R})$, let $\phi(n) = A^n$.)
Group homo $\phi: \mathbb{Z} \to \text{GL}_d(\mathbb{R})$

(i.e., $\phi(m + n) = \phi(m) \cdot \phi(n)$)

$\not\Rightarrow$ extends to homo $\hat{\phi}: \mathbb{R} \to \text{GL}_d(\mathbb{R})$.

Because: maybe $\det(\phi(1)) < 0$.

However, $\det(\phi(\text{even})) > 0$.

\[
\begin{align*}
\det(\phi(2m)) & = \det(\phi(m + m)) \\
& = \det(\phi(m) \cdot \phi(m)) \\
& = \left(\det(\phi(m))\right)^2 \\
& > 0.
\end{align*}
\]
May have to ignore odd numbers: restrict attention to even numbers.

Analogously, may need to restrict to multiples of 3 (or 4 or 5 or \ldots )

Restrict attention to multiples of $N$

\{multiples of $N$\} is a subgroup of $\mathbb{Z}$

“ Restrict attention to a finite-index subgroup”
Prop. Group homomorphism $\phi : \mathbb{Z}^k \to \text{GL}_d(\mathbb{R})$

$\Rightarrow \phi$ “almost” extends to homo $\hat{\phi} : \mathbb{R}^k \to \text{GL}_d(\mathbb{R})$
such that $\hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$. (“Zariski closure”)

This means $\mathbb{Z}^k$ is superrigid in $\mathbb{R}^k$.

“Homomorphisms defined on $\mathbb{Z}^k$ almost extend to be defined on $\mathbb{R}^k$”

Generalize to nonabelian groups.
Lagrange interpolation:

there is a polynomial curve

\[ y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \]

through any \( n + 1 \) points.
Idea: Zar closure is like *convex hull*.

Image of $\phi$ controls image of $\hat{\phi}$.

*Eg.* If all matrices in $\phi(\mathbb{Z})$ commute, then all matrices in $\hat{\phi}(\mathbb{R})$ commute.

*Eg.* If all matrices in $\phi(\mathbb{Z})$ fix a vector $v$, then all matrices in $\hat{\phi}(\mathbb{R})$ fix $v$. 
(3. The analogy)

**Combinatorial superrigidity:**

Make a copy of the object, according to the combinatorial rules.

The copy is the exact same shape as the original.

Maybe not exactly the same object:
- may be rotated from the original position;
- may be translated from original position.

These are trivial modifications: rotations and translations are symmetries of the whole universe (Euclidean space $\mathbb{R}^3$).
Combinatorial superrigidity:

Make a copy of the object, according to the combinatorial rules.

The same result can be obtained by keeping the original object and moving the whole universe to a new position.

“If the object can be moved somewhere, then the whole universe can be moved there.”

Let $H$ be a subgroup of a group $G$.

*Superrigid* means:

homomorphism $\phi: H \to \text{GL}_d(\mathbb{R})$ extends to homomorphism $\hat{\phi}: G \to \text{GL}_d(\mathbb{R})$

Group-theoretic superrigidity:

Make a copy of $H$ as a group of matrices.

The same copy of $H$ can be obtained by moving all of $G$ into a group of matrices.
4. Superrigid subgroups

**Prop.** Group homomorphism \( \phi: \mathbb{Z}^k \to \text{GL}_d(\mathbb{R}) \)

\( \Rightarrow \) \( \phi \) “almost” extends to homo \( \hat{\phi}: \mathbb{R}^k \to \text{GL}_d(\mathbb{R}) \)

such that \( \hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}. \) (“Zariski closure”)

This means \( \mathbb{Z}^k \) is superrigid in \( \mathbb{R}^k \).

**Generalize to nonabelian groups.**

\( \mathbb{Z}^k \) is a lattice in \( \mathbb{R}^k \). I.e.,

- \( \mathbb{R}^k \) is a (simply) connected grp ("Lie group")
- \( \mathbb{Z}^k \) is a discrete subgroup
- all of \( \mathbb{R}^k \) is within a bounded distance of \( \mathbb{Z}^k \)

\[ \exists C, \ \forall x \in \mathbb{R}^k, \ \exists m \in \mathbb{Z}^k, \ \ d(x, m) < C. \]

\( H \) is a lattice in \( G \)
All of $\mathbb{R}^k$ is within $\sqrt{k}/2$ of $\mathbb{Z}^k$
Let us consider solvable groups.

A connected subgroup $G$ of $\text{GL}_d(\mathbb{C})$ is solvable if it is upper triangular

$$G \subset \begin{pmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \end{pmatrix}$$

(or is after a change of basis).

Eg. All abelian groups are solvable.

Proof. Every matrix can be triangularized over $\mathbb{C}$. Pairwise commuting matrices can be simultaneously triangularized.
**Prop.** $H$ superrigid in $G$

$$\Rightarrow \overline{H} = \overline{G} \pmod{Z(G)}.$$ 

**Proof.** The inclusion $H \hookrightarrow \text{GL}_d(\mathbb{R})$

must extend to $G \hookrightarrow \text{GL}_d(\mathbb{R})$

with $G \subset \overline{H}$.

Converse:

**Thm** (Witte). *A lattice $H$ in a solvable grp $G$ is superrigid iff* $\overline{H} = \overline{G} \pmod{Z(G)}$. 
Examples of lattices.

\[ G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ G = \begin{pmatrix} \mathbb{R}^+ & 0 & 0 \\ 0 & \mathbb{R}^+ & 0 \\ 0 & 0 & \mathbb{R}^+ \end{pmatrix} \]

\[ H = \begin{pmatrix} 2\mathbb{Z} & 0 & 0 \\ 0 & 2\mathbb{Z} & 0 \\ 0 & 0 & 2\mathbb{Z} \end{pmatrix} \]
\[ G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \bar{G} = G \]

\[ H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z} i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \bar{H} = G \]

\[ G' = \begin{pmatrix} 1 & t & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix} \]

\[ \bar{G}' = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{T} \end{pmatrix} \]

\[ H' = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z} i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = H \]
$H$ is a lattice in both $G$ and $G'$. 

$$
\overline{H} = G \neq \overline{G}' \quad \text{so} \quad \overline{H} \neq \overline{G}'
$$

$H$ is \textit{not} superrigid in $G'$. 

E.g., the identity map $\phi: H \rightarrow H$ does not extend to homo $\hat{\phi}: G' \rightarrow \overline{H}$.

\textit{Proof.} Note that $\overline{H} = G$ is abelian but $G'$ is not abelian.

A nonabelian group cannot be embedded in an abelian one.
\[ H \neq G' \]: some of the rotations associated to \( G' \) do not come from rotations associated to \( H \)

\[
\text{rot} \left( \begin{array}{cc} \alpha & \ast \\ 0 & \beta \end{array} \right) = \left( \begin{array}{cc} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\beta}{|\beta|} \end{array} \right)
\]


